International Journal of Production Research

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tprs20

Dynamic compensation and contingent sourcing strategies for supply disruption

Shanshan Li, Yong He & Stefan Minner

To cite this article: Shanshan Li, Yong He & Stefan Minner (2021) Dynamic compensation and contingent sourcing strategies for supply disruption, International Journal of Production Research, 59:5, 1511-1533, DOI: 10.1080/00207543.2020.1840643

To link to this article: https://doi.org/10.1080/00207543.2020.1840643

INTERNATIONAL JOURNAL OF PRODUCTION RESEARCH
2021, VOL. 59, NO. 5, 1511–1533
https://doi.org/10.1080/00207543.2020.1840643

Dynamic compensation and contingent sourcing strategies for supply disruption

Shanshan Li [a], Yong He [b] and Stefan Minner [c]

[a]School of Finance, Nanjing Audit University, Nanjing, People's Republic of China; [b]School of Economics and Management, Southeast University, Nanjing, People's Republic of China; [c]TUM School of Management, Technical University of Munich, Munich, Germany

ABSTRACT

Alternative measures to deal with supply disruptions exist. We consider a make-to-order (MTO) supply chain with one manufacturer who sources from a single supplier. When a supply disruption occurs, the manufacturer can choose to satisfy some demand by either maintaining production through safety stocks or through a secondary contingent source, and turn some unmet demand into backorders on the basis of compensation. An optimal control model under consideration of the customers' dynamic reactions to the joint implementation of these strategies is formulated with the objective of minimising the cost of disruption. Through the application of Pontryagin's Maximum Principle, optimal mitigation strategies are established in closed form. They provide analytical guidance on how to dynamically and jointly adapt the quantity of contingent sourcing, the price of compensation, and the speed of safety inventory consumption. The results indicate how cost and time-related factors impact these strategies. We also demonstrate that pure strategies are only effective in tackling short supply shortages. For long disruptions, it is superior to adopt combined strategies that simultaneously incorporate two countermeasures in certain periods.

ARTICLE HISTORY
Received 3 March 2020
Accepted 12 October 2020

KEYWORDS
Supply disruption; contingent sourcing; compensation; safety inventory; control theory

1. Introduction

In recent years, with trends like global purchases and lean production, the risk of supply chain disruption has increased and the impact of disruptions can be substantial. An empirical investigation that polled 151 supply chain managers found that 73% of the companies experienced costly disruptions in the last five years (Schmitt et al. 2017). In 2020, due to the epidemic Covid-19, 94% of the Fortune 1000 companies suffer from disruption to their supply chains (Fortune 2020). Academics and practitioners have paid considerable attention to the management of supply disruptions. A large number of strategies that aim at alleviating the negative impact caused by failures in the supply process have been proposed, for instance, demand switching (Tomlin 2009), inventory policies (Qi 2013), compensation to customers (Chen et al. 2015), contingent sourcing (He et al. 2020), supplier diversification (Silbermayr and Minner 2014; 2016; Golmohammadi and Hassini 2020), dynamic scheduling (Ivanov, Dolgui, and Sokolov 2018), recovery (Ivanov et al. 2017), and others.

In practice, safety inventory hedging is one of the most applied proactive strategies for handling supply disruptions (Gao 2015). Nonetheless, the hedging of supply shocks by means of pure safety stock normally requires a large amount of stock for a long time. Consequently, it generates high inventory costs, and can jeopardise a firm's long-term profitability and viability under some situations, especially when disruptions are rare events. To avoid these costs, many firms are strongly tempted towards cutting back the scale of needed stock.

Particularly in a make to order (MTO) supply chain, companies tend to keep minimum inventories (of materials, components and parts in buffer, but not of a finished product) and adopt just-in-time (lean) practices ranging from procurement to the delivery of final products (Xu 2020). For example, Dell Computers, which uses MTO production, only carries a maximum of 72 h inventory across its entire operation (Breen 2011). As a result, a minor unpredictable supply shortage might lead to a production halt and devastating losses. For an MTO production system that carries insufficient preventive inventory, it is essential to properly incorporate other proactive or reactive policies as a supplement to effectively manage supply.

CONTACT Yong He hy@seu.edu.cn School of Economics and Management, Southeast University, Nanjing, 210096, People's Republic of China

This article has been corrected with minor changes. These changes do not impact the academic content of the article

Supplemental data for this article can be accessed here. https://doi.org/10.1080/00207543.2020.1840643

The importance of developing mixed strategies by combining diversified proactive policies or incorporating both proactive and reactive countermeasures attracts increasing attention in recent studies (Ivanov et al. 2016; Topan and van der Heijden 2020). In reality, many companies also tend to adopt mixed strategies. For instance, to avoid devastating disruptions, Huawei (a leading global provider of information and communications technology (ICT) infrastructure and smart devices in China) incorporates supplier selection and safety inventory from the proactive aspect (UKEssays 2018). Nonetheless, the research is still in its infancy incorporating proactive and reactive countermeasures, especially taking into consideration the implementation of both supply-side and demand-side tools in the progress of realising reactive policies, where the customers' post-disruption reaction is sensitive to both price and time. Thus, to fill this gap, we consider an MTO production system with one manufacturer who sources from a single supplier where the supplier has already faced a disruption. The manufacturer carries a certain amount of raw materials as a precaution. After the occurrence of a disruption, the existing safety stock of raw materials can be gradually consumed to continue production. The market demand is deterministic but sensitive to the waiting time for delivery. In terms of reactive policies, we consider both demand-side compensation and supply-side sourcing.

Contingent sourcing is a widely utilised emergency countermeasure where an enterprise turns to a backup supplier (or to the spot market) in the event of a failure at its main supplier (Saghafian and Van Oyen 2016). As pointed out by Rigby and Bilodeau (2009), 63% of their surveyed companies employed this strategy in 2008. However, the contingent source normally comes with a high price and a lead time. For example, due to the supply disruption caused by Covid-19, the Indian automobile manufacturers now adopt an emergency procurement of components from China and South Korea, paying a high price on chartered flights (Yadav 2020).

Compensation is another common reactive mechanism used for tackling the customers' dissatisfaction during an empty-stock period. In 2011, when HP and its partners failed to fulfil the orders of TouchPad in time, its retailers employed multiple forms of compensation to reduce the damage of the stock-out, including apologies, a free service for future deliveries, future discounts, etc. (Dong et al. 2015). The provided compensation level (discount price) is commonly related to the length of the customers' waiting time. Inspired by the long disruption caused by Covid-19, both researchers and managers in industries emphasise the importance of establishing strategies to reduce the damage from the demand-side (Accenture 2020).

Motivated by the above examples, we consider a lead time for contingent sourcing. The compensation is designed by the manufacturer as a dynamic discount based on the customers' waiting time, which is determined by the customers' arrival time and the time when the purchase is fulfilled. In view of constrained production capacity, customers who choose to wait will receive products at a later point in time after the supply disruption ends, following a 'First come first served' principle.

We examine the joint dynamic decision of compensation pricing, contingent sourcing, and inventory consumption for a manufacturer who wishes to optimally mitigate supply disruption at any point in time during a supply shortage. By presenting the results in closed form, our paper provides two main contributions to the literature on supply disruption management. First, based on the control theory, we develop a new approach for jointly designing optimal supply disruption-management strategies, capturing the customers' state-dependent behaviour. The state dynamics of backlogged demand and inventory under the joint countermeasures performed by manufacturers are captured in the disruption duration and recovery periods. Second, we propose optimal time-dependent mitigation strategies that jointly incorporate proactive and reactive supply- and demand-side aspects. The established strategies provide analytical guidance for manufacturers on how to optimally adjust the joint time-dependent decision by contingent sourcing, the compensation level, and inventory consumption. The results also indicate the sequence of announcing countermeasures to customers in the process of realising combined strategies. We identify how cost factors and time-related factors (customer sensitivity to time, disruption duration, and lead time of sourcing) play different roles during the design of mitigation strategies. Our results provide both analytical and numerical evidence that it is essential to develop dynamic combined strategies for hedging against disruptions.

The remainder of this paper is organised as follows. In Section 2, a literature review is presented. The problem description and model formulation are given in Section 3. Section 4 proposes the optimal dynamic strategies for mitigating a supply shortage. Managerial insights and conclusions are presented in Sections 5–6.

2. Literature review

A growing body of literature studies the management of risks in supply chains. Snyder et al. (2016) and Baryannis et al. (2019) provide detailed discussions in this field. Our work is mainly related to two streams: pure strategies (i.e. contingent sourcing, customer compensation, and safety inventory) and combination strategies.

2.1. Pure strategies

A buyer can utilise many procurement strategies to proactively or reactively manage supply disruptions, for instance, backup sourcing (with pre-disruption contracts), contingent sourcing, and supplier diversification/selection with the splitting of orders among multiple/dual suppliers to proactively reduce the disruption risk or to reactively replenish inventory (Namdar et al. 2018; He et al. 2019; Gupta and Ivanov 2020). We consider the contingent (backup/dual) sourcing strategy with no underlying contracts: The manufacturer/retailer orders from a single primary supplier, and will place an emergency order with a secondary supplier (spot market or backup) when the primary supplier fails to deliver. This strategy is studied by Tomlin (2006) and other subsequent papers.

Extensive work investigates the optimal time and quantity of using the second source. Considering a supply chain with two competing manufacturers, Gupta, He, and Sethi (2015) study several firms' buying decisions and reveal that supply disruption and procurement times jointly impact the optimal order quantities. Recent research on contingent sourcing strategies concerns dynamic aspects. Kouvelis and Li (2012) investigate the ex-post dynamic emergency response that identifies the timing and size of an emergency order that needs to be placed, and find that the response is cost-effective when the coefficient of variation of the uncertain lead-time is high. Saghafian and Van Oyen (2016) investigate the optimal design of flexibility in a backup system under dynamic disruption risks. Based on a post-disruption customer behaviour forecast, He et al. (2020) propose dynamic contingent sourcing strategies. The existing literature further investigates the benefits of contingent sourcing through comparison. By developing coordination models under both uncertain supply production and uncertain demand, Chen and Yang (2014) compare emergency backup sourcing decisions under various scenarios of relative channel power and reveal that the decentralised operation is rather dependent on backup sourcing products. Looking at two competing buyers who use either emergency sourcing or an optimal allocation procurement strategy, He, Huang, and Yuan (2016) compare these two strategies and point out that the procurement decision is critically determined by supplier reliability and sales price.

The relevant literature regarding safety inventory strategies mostly focuses on examining the optimal stock level under different kinds of supply uncertainty, or on the order placement under consideration of safety stock. Using an EOQ model that considers random disruptions at both the supplier and the retailer end, Qi, Shen, and Snyder (2009) explore the optimal order size for the retailer and examine the safety stock level the retailer needs in order to protect against supplier disruptions. Some research extends to other forms of uncertainty or limitation when analysing an inventory holding strategy. For example, taking time-dependent costs for both inventory holding and shortages into consideration, Taleizadeh (2017) develops a lot-sizing inventory model that hedges against supply disruptions. Later, based on the same assumption about cost, Saithong and Luong (2019) propose a periodic-review base-stock inventory policy in a two-stage supply chain. Svoboda, Minner, and Yao (2020) provide a comprehensive overview of the existing multi-supplier inventory models.

In the context of safety inventory, the works most relevant to our paper are the ones by Paul and Rahman (2018) and Topan and van der Heijden (2020). Instead of optimising the pre-disruption inventory or post-disruption replenishment, they focus on how to reactively consume reserved stock. Using a simulation model, Paul and Rahman (2018) generate a recovery plan based on safety stock to overcome sudden supply delays. In a multi-item two-echelon spare parts supply network, Topan and van der Heijden (2020) investigate operational safety inventory interventions from both proactive (to reduce stock-out risks) and reactive aspects (to fulfil a demand that is not satisfied), including lateral transshipments between local warehouses, emergency shipments from a central warehouse, and passively waiting.

About the pure compensation strategy, Bhargava, Sun, and Xu (2006) state that stock-out compensation is broadly implemented in traditional and online retailing. Most of the existing work defines compensation as a price reduction for customers who must wait and will be satisfied later. Chen et al. (2015) propose another compensation mechanism for backorders, namely a priority auction with an admission price. Under the auction compensation, priority is allocated according to the customers' bid prices. Recently, a few related studies extend a general pricing problem in response to supply uncertainties (Gupta, Ivanov, and Choi 2020).

2.2. Combination strategies

In the second stream, a few types of combination strategies are proposed via comparison. Based on a single-product setting where one supplier is unreliable and

Table 1. The comparison of existing literature.

Papers	SI	CS	P	Approch	Recovery	Time-dependent	Joint Decision	Specification
Gupta, He, and Sethi (2015)		√		GT				
Kouvelis and Li (2012)	√	√		QM		√		Lead time
Saghafian and Van Oyen (2016)	√	√		DP	√	√		
He et al. (2020)	√	√		QM	√	√		
Chen and Yang (2014)	√	√		GT				coordination
He, Huang, and Yuan (2016)		√	√	GT				
Qi, Shen, and Snyder (2009)	√			GT				
Taleizadeh (2017)	√			QM		√		
Saithong and Luong (2019)	√			QM		√		
Paul and Rahman (2018)	√			QM	√			
Topan and van der Heijden (2020)	√			MIP		√	√	lateral transshipments/ emergency shipments
Bhargava, Sun, and Xu (2006)	√		√	GT			√	
Chen et al. (2015)	√		√	GT			√	First-come-first served
Tomlin (2006)	√	√		GT				
Su and Zhang (2009)	√		√	GT				
Li, He, and Chen (2017)	√	√		QM	√			Lead time
Shao (2018)			√	GT		√	√	redundant capacity sharing
Kumar, Basu, and Avittathur (2018)		√	√	GT			√	
Wang and Yu (2020)		√	√	GT			√	
Gupta, Ivanov, and Choi (2020)			√	GT				
Azad and Hassini (2019)	√		√	MIP	√	√	√	outsourcing
Our paper	√	√	√	CT	√	√	√	First-come-first served &Lead time

GT: Game theory; QM: Quantitative model; DP: Dynamic programming mixed-integer; MIP: mixed-integer programming; CT: Control theory. SI: Safety Inventory; CS: Contingent Sourcing; *P*: Pricing or Compensation.

another is reliable but more expensive, Tomlin (2006) proposes a mixed strategy that incorporates partial sourcing and inventory. Referring to ex-post compensation as an availability guarantee, Su and Zhang (2009) compare it with an ex-ante strategy that deals with stock-outs: commitment guarantee (commits to a particular quantity) combined with commitment and availability guarantees first-best outcomes for the seller. By comparing emergency backup sourcing, production recovery, and passive acceptance, Li, He, and Chen (2017) propose two types of combination strategies, respectively for prevention and non-prevention systems.

Shao (2018) provides a framework for investigating joint optimisation based on two policies that mitigate production disruptions: compensation and redundant capacity sharing. Based on a duopoly setting, Kumar, Basu, and Avittathur (2018) explore how a retailer can jointly use pricing decisions and contingent sourcing to cope with the supply disruption risk when there is another competing retailer with a more reliable supply. The emergency sourcing quantity and sales price are simultaneously examined. Similar to this work, considering a committed and a responsive pricing strategy under which the retailer adjusts the sales price before and after the supply state is realised, Wang and Yu (2020) investigate whether a supply-side contingent sourcing strategy should be jointly adopted with a demand-side pricing strategy to mitigate supply disruption. Considering dynamic pricing as a recovery lever to manage demand during supply disruptions, and under the assumption that demand can be satisfied via a partially disrupted production facility or outsourcing, Azad and Hassini (2019) develop a portfolio of recovery strategies by a programming model and a Benders decomposition algorithm. The strategies incorporate pricing, transportation rerouting, and outsourcing. Two uncertain recovery parameters are considered via a scenario approach: the number of disrupted capacities in the facilities and the length of the recovery period.

Based on the above analysis, our paper differs from the existing literature in four dimensions (shown in Table 1). First, in the context of reactive supply disruption mitigation, several recent papers initially realise the importance of incorporating demand-side strategies along with supply-side sourcing strategies, where the time-independent (static) pricing is considered as an endogenous demand-adjusting tool. Taking the customers' finite tolerance to wait time into account, we consider that demand is both price-sensitive and time-sensitive. A dynamic pricing (compensation) decision is incorporated. Second, unlike most of the literature that only focuses on the disruption duration period, we consider the post-disruption recovery period as well when evaluating disruption loss. The recovery period is characterised based on two aspects: the production capacity and the backlogged demand that has accumulated during disruption and will be satisfied after the end of the disruption under a 'first-come-first-served' principle. Third, in addition to a high price, the lead time of contingent sourcing is also considered. Together with the customers' finite tolerance to waiting time, we capture the possibility that a contingent source with a long lead time might be incapable of satisfying demand. Fourth, different from most work that applies control theory responding to uncertain disturbances in a system (Ivanov and Dolgui 2019), we focus on specific strategies in closed form. By incorporating proactive inventory consumption, reactive supply-side sourcing, and demand-side compensation, the combination strategy is optimised with the addition of a dynamic joint decision.

3. Problem description and model formulation

A firm produces and sells a single product to customers, and sources from a regular supplier who is unreliable and has infinite capacity. There is an emergency supplier who is reliable, expensive, capacity constrained, and a lead time t_0. The normal demand rate of finished products is deterministic and, without loss of generality, normalised to '1'. The firm practices make-to-order manufacturing with a production capacity $P_C > 1$, which implies that no inventory of finished products is kept on hand. Before the appearance of a supply disruption, the production is realised at the demand rate '1'. In view of the unreliability of the regular supplier, the manufacturer proactively carries I_0 units of safety stock in raw materials to support the option of responding to shortages caused by supply disruptions. We assume that a supply disruption occurs at time '0' and has a deterministic duration of T periods.

Production stops immediately if no countermeasures are adopted, and thus a stock-out occurs for customers who arrive at any time t. Customers behave in two ways: they either leave (lost sales) or stay (backorders). To avoid lost sales and thus to effectively mitigate supply disruption, the manufacturer simultaneously makes two decisions: control $r_2(t)$ as backorders through compensation, and satisfy $d(t)$ by resuming production through the contingent source and safety inventory. As a result, $1 - r_2(t) - d(t)$ demand is lost at time t, and backlogged demand $b(t)$ is accumulated at the rate $\dot{b}(t) = r_2(t)$.

Therefore, two questions are crucial for the manufacturer at time t: (1) How much demand should be kept as backorders (to be satisfied after the end of the supply disruption)? How much demand should be met during the supply shortage? That is, determining the optimal dynamic joint decision $\{d(t), r_2(t)\}$ to minimise disruption losses. (2) how much compensation should be provided? how many units should be procured using a contingent source and how much safety stock should be consumed simultaneously?

Table 2. Notation.

	Notations	Description
Decisions	$s(t)$	number of customers who are provided contingent sourcing
	$p(t_{fulfilled}, t)$	compensation provided to customers
	$r_2(t)$	rate of backorder demand
	$d(t)$	quantity of satisfied demand during disruption
Parameters	T	length of disruption duration
	t_0	lead time of the second source
	$t_{fulfilled}$	time when backlogged demand is fulfilled.
	t_{wait}	waiting time before backlogged demand is fulfilled.
	t_{I0}	time when inventory reaches zero
	θ	customer sensitivity to waiting time
	$r_0(t)$	customers' willingness to place backorder without compensation
	r_1	rate of customers who accept contingent sourcing, $r_1 = 1 - \theta t_0$
	P_C	production capacity of the manufacturer, $P_C > 1$
	$b(t)$	quantity of the backlogged demand
	$\dot{b}(t)$	marginal increase of backlogged demand.
	I_0	initial amount of safety raw materials, $I_0 < T$.
	$I(t)$	inventory level
	$\dot{I}(t)$	marginal increase of inventory.
	c_0	unit sourcing cost from the regular supplier
	c_s	unit contingent sourcing cost from the emergency supplier
	c_1	unit production cost
	c_h	unit inventory holding cost per unit of time
	c_p	unit cost for each level of compensation (normalised discount)
	c_l	unit cost for a lost sale

For convenience, we introduce the following auxiliary variables

(i) $f_1 = b^*(\tau_1)$, $f_2 = \frac{P_C-1}{\theta}(2 - P_C - \theta T + \theta t) + C_1 e^{-\frac{\theta t}{P_C-1}}$, $f_3 = b^*(\tau_3) + t - \tau_3$, $f_4 = b^*(\tau_4) + (1 - r_1)(t - \tau_4)$.

(ii) $F_1 = C_{F1} e^{-\frac{\theta t}{P_C-1}} + c_0 - \lambda_2(0) + c_h\left(\frac{P_C-1}{\theta} - t\right)$,

$F_2 = C_{F2} e^{-\frac{\theta t}{P_C-1}} + c_0 + c_1 - c_l$, $F_3 = C_{F3} e^{-\frac{\theta t}{P_C-1}} + c_0 - \lambda_2(0)(1 - r_1) - c_s r_1 + c_h(1 - r_1)\left(\frac{P_C-1}{\theta} - t\right)$,

$F_4 = \lambda_1(\tau_{F4}) - \frac{\theta(t-\tau_{F4})}{P_C-1} c_p$.

(iii) $M_1 = c_0 + c_1 - c_l$, $M_2 = c_0 - c_s$.

The τ_k and τ_{F4} respectively present the entry points of the time intervals with $b^*(t) = f_k$ and $\lambda_1 = F_4$. Where, $k = 1, 2, 3, 4$. The constants C_1, C_{F1}, C_{F2} and C_{F3} are to be determined by the continuity of the backlogged demand and the corresponding co-state variable λ_1.

We assume $c_1 + c_s < c_l$, which means that it is preferable to procure contingent sources to resume production to doing nothing and suffering lost sales Table 2.

Next, we start by exploring the dynamics of inventory and demand under the combined countermeasure incorporating compensation, sourcing, and inventory consumption.

3.1. Dynamics of inventory and demand during and after disruption

3.1.1. The dynamics of the compensation policy

To quantify how many of a group of customers are willing to place a backorder, the backorder rate is well-utilised in quite a few publications on stock-out management.

Under compensation, customers who backorder will receive the product with a delay and with a compensation $p(t_{fulfilled}, t)$ at time $t_{fulfilled}$ after the end of the disruption. When offered such compensation in times of stock-out, some customers are willing to accept and postpone their order. We assume that the customers' willingness to back-order is proportional to the level of the provided incentive, for instance, the size of a price discount (Drake and Pentico 2011). On the other hand, some customers are willing to wait for later fulfilment without incentives because of a good reputation of the supplier or due to brand loyalty (Sarkar, Mandal, and Sarkar 2015). Thus, in alignment with the common assumption (Ding, Kouvelis, and Milner 2006), we allow a nonnegative back-ordering willingness $r_0(t)$ if no compensation is offered, and model the customers' back-ordering willingness $u(t)$ under compensation as

$$u(t) = r_0(t) + p(t_{fulfilled}, t). \tag{1}$$

The compensation level $p(t_{fulfilled}, t)$ is a normalised price discount. $u(t) = 1$ when $p(t_{fulfilled}, t) = 1 - r_0(t)$. In other words, all customers are willing to backorder if the compensation level reaches $1 - r_0(t)$. From the perspective of the manufacturer, there is no need to provide compensation that exceeds this level. Hence, the compensation level is limited by $0 \leq p(t_{fulfilled}, t) \leq 1 - r_0(t)$.

Most existing studies assume that the backorder rate without incentives decreases over the lead time of delivery in the form of an exponential, polynomial, or linear function (Shao and Dong 2012; Pentico, Toews, and Drake 2015). We consider a linear form of $r_0(t)$ in this study. Therefore, by limiting $0 \leq r_0(t) \leq 1$,

$$r_0(t) = [1 - \theta(t_{fulfilled} - t)]^{+}. \tag{2}$$

The length of the waiting time is given as $t_{wait} = t_{fulfilled} - t$. All customers are willing to backorder without compensation if the sensitivity is zero. Such a case rarely occurs in reality, thus we focus on $0 \leq r_0(t) < 1$. That is, the partial back-ordering behaviour introduced in (2) will happen in one of the following two states.

(i) **Demand State 1** where $r_0(t) = 0$: $t_{wait} \geq 1/\theta$. If the waiting time exceeds a critical length or customers are significantly sensitive, no customer is willing to delay the purchase without compensation.

(ii) **Demand State 2** where $0 < r_0(t) < 1$: $t_{wait} < 1/\theta$. Due to the customers' patience or the short waiting time, a positive fraction of customers might choose to backorder without compensation.

Based on the customers' back-ordering willingness $u(t)$, the manufacturer decides upon the backorder rate $r_2(t)$ and the compensation level $p(t_{fulfilled}, t)$. Note that the number of customers that actually backorder is limited by the number of customers who are willing to backorder, that is, $r_2(t) \leq u(t)$.

3.1.2. The dynamics of demand and inventory to contingent sourcing

To satisfy $d(t)$ during disruption, the manufacturer can choose between two options to resume production: inventory and contingent sourcing. The difference between these two policies is that customers receive products immediately if safety inventory is used, whereas they can only be satisfied after a lead time if contingent sourcing is adopted. As a result, if the lead time t_0 of contingent sourcing exceeds a tolerable length of time, some customers might reject the contingent sourcing policy offered by the manufacturer, which results in lost sales.

Suppose that the contingent sourcing policy is offered to $s(t)$ customers, $0 \leq s(t) \leq 1$. Due to the lead time, $r_1 = 1 - \theta t_0$ of them accept it and will be satisfied at time $t_0 + t$. To fulfil this portion of the demand, the manufacturer thus procures $r_1 s(t)$ raw materials at a price of c_s. In the meantime, inventory is consumed at the rate of $-\dot{I}(t)$, where $\dot{I}(t) \leq 0$. Therefore, by using these two channels of materials, production is realised at the rate of $r_1 s(t) - \dot{I}(t)$ to satisfy $d(t)$, i.e. $d(t) = r_1 s(t) - \dot{I}(t)$.

3.1.3. The dynamics of demand and inventory after disruption

Due to the joint adoption of the above policies, a certain amount of backlogged demand has accumulated at the end of the supply disruption, and the safety stock is completely consumed. Therefore, we take into account a time interval (T, T_{max}), namely the recovery period. During recovery, the regular supplier restores, hence raw materials are procured at a cheap price and production can be resumed and safety stock can be refilled. The safety stock of raw materials is refilled after the backlogged demand is completely met at the time T_{max}. Furthermore, in view of the infinite capacity of the regular supplier, we assume that the inventory instantly reverts to I_0 at T_{max}.

To satisfy both the backlogged demand and the real-time demand, production is realised at the maximum rate. The backlogged demand decreases at the rate of $P_C - 1$ during the recovery period. On the other hand, in a real market scenario, backlogged demand is commonly

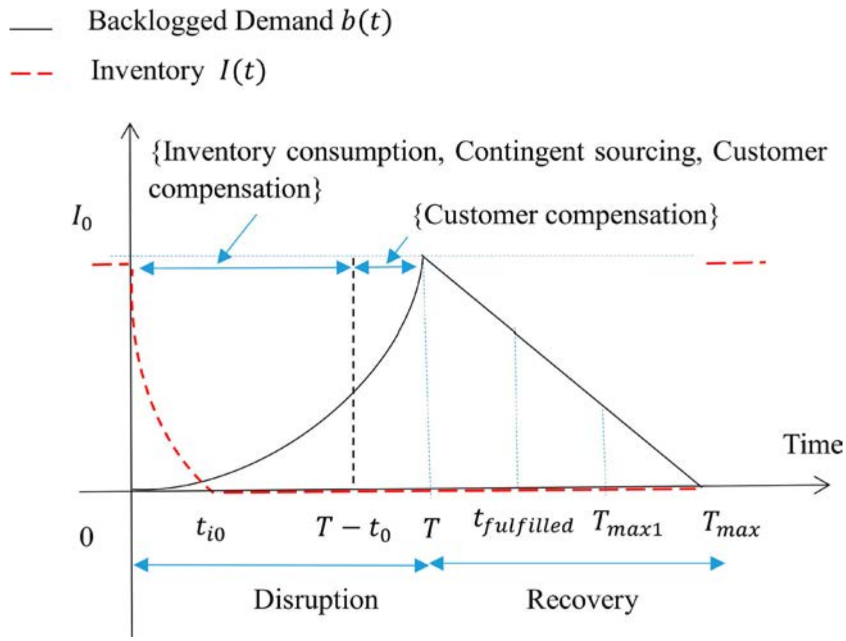


Figure 1. Inventory and backlogged demand in the MTO system.

served in the order of arrival. That is, for the customer who backorders at time t, the time when she/he is chosen based on a 'First come first served' principle (Chen et al. 2015), can be derived as

$$t_{fulfilled} = T + \frac{b(t)}{P_C - 1}. \tag{3}$$

Given (3), the length of the customers' waiting time is specified, and the customers' partial backordering behaviour $r_0(t)$ (see (2)) can be further determined in Lemma 1.

Lemma 1: *Under the 'First come first served' principle, the customers' partial backordering willingness without compensation is derived as $r_0(t) = (1 - \theta t_{wait})^+$. Where,*

$$t_{wait} = T + \frac{b(t)}{P_C - 1} - t, \tag{4}$$

and t_{wait} increases over time if $r_2(t) < P_C - 1$ and decreases otherwise.

Proof: According to (3), $t_{wait} = t_{fulfilled} - t = T + \frac{b(t)}{P_C-1} - t$. The derivative of t_{wait} with respect to the time variable t is derived as $\frac{dt_{wait}}{dt} = \frac{db(t)/dt}{P_C-1} - 1 = \frac{r_2(t)}{P_C-1} - 1$. The Lemma 1 is achieved. ∎

Lemma 1 illustrates the dynamics of the customers' state-dependent behaviour. If $b(t) \geq \left(\frac{1}{\theta} + t - T\right)(P_C - 1)$, too much backlogged demand has accumulated in the system, thus no customer is willing to wait in line without any incentives, i.e. $r_0(t) = 0$. Otherwise, the customers' behaviour falls into Demand State 2 where $r_0(t) > 0$. Furthermore, the results show an important property of the customers' dynamic behaviour. At time t, if the manufacturer decides to accept a backorder rate that exceeds his/her recovery capability during the recovery period, it will reduce the willingness of subsequent customers to wait without compensation.

The dynamics of safety inventory and backlogged demand during $(0, T_{max})$ in this MTO production system are depicted in Figure 1.

Due to lead time t_0, raw materials obtained through contingent sourcing in the time interval $(T - t_0, T)$ can only be utilised for production after the supplier is restored at time T (see Figure 1). Thus, there is no need to implement contingent sourcing during this short phase. In other words, the only available countermeasure in $(T - t_0, T)$ is customer compensation. Considering that fruitful research has been done on pure pricing or compensation, we focus on the joint implementation of inventory consumption, contingent sourcing, and customer compensation to cope with the disruption during the time interval $(0, T - t_0)$.

As afore analysed, the backlogged demand accumulated in $(0, T - t_0)$ will be met during (T, T_{max1}). That is, both the disruption duration $(0, T - t_0)$ and the disruption recovery period (T, T_{max1}) are impacted, where $T_{max1} = T + b(T - t_0)/(P_C - 1)$ (see (3)). Next, to identify the optimal decision for mitigating disruption, we evaluate the disruption impact incurred during these two periods.

Table 3. Strategies overview.

Decisions						
s^*	$r_2{}^*$	p^*	d^*	$-\dot{I}^*$	b^*	Strategies
0	0	0	1	1	f_1	I_H
0	$r_0{}^*$	0	$1-r_0{}^*$	$1-r_0{}^*$	f_2	I_L
1	0	0	1	$1-r_1$	f_1	SI_H
1	$r_0{}^*$	0	$1-r_0{}^*$	$1-r_0{}^*-r_1$	f_2	SI_L
1	0	0	r_1	0	f_1	S_H^0
1	$r_0{}^*$	0	r_1	0	f_2	S_H^r
$(1-r_0{}^*)/r_1$	$r_0{}^*$	0	$1-r_0{}^*$	0	f_2	S_L
0	1	1	0	0	f_3	P_H
0	1	$1-r_0{}^*$	0	0	f_3	P_L
1	$1-r_1$	$1-r_1$	r_1	0	f_4	SP_H
1	$1-r_1$	$1-r_1-r_0{}^*$	r_1	0	f_4	SP_L

3.2. Disruption impact

Based on the demand and inventory dynamics during the impacted periods $(0, T-t_0) \cup (T, T_{max1})$, the disruption impact represented by the difference between the cost with and the cost without disruption, is derived as

$$\Delta C = C_{1D} - C_{1N} + \Delta C_2. \tag{5}$$

The first term $C_{1D} = \int_0^{T-t_0} \{c_1 d(t) + c_s r_1 s(t) + c_h I(t) + c_p p(t_{fulfilled}, t) + c_l [1 - r_2(t) - d(t)]\} dt$ gives the cost incurred during $(0, T-t_0)$, including the production cost, the contingent sourcing cost, the inventory holding cost, the compensation cost, and the lost-sale cost. The second term $C_{1N} = \int_0^{T-t_0} (c_0 + c_1 + c_h I_0) dt$ gives the cost during $(0, T-t_0)$ with no disruption, including the production cost, the regular sourcing cost, and the inventory holding cost. Therefore, the difference between these two items identifies the disruption cost during $(0, T-t_0)$. The third term $\Delta C_2 = (c_0 + c_1) \int_0^{T-t_0} r_2(t) dt + c_0 I_0$ represents the disruption cost incurred during (T, T_{max1}), including the production cost of satisfying backlogged demand $b(T-t_0)$ and the recovery cost of refilling safety inventory.

3.3. Mathematical representation of optimal mitigation strategies

The post-disruption state dynamics of backlogged demand and inventory in the production-inventory system are characterised as $\dot{b}(t)$ and $\dot{I}(t)$. $r_2(t)$, $s(t)$, $p(t_{fulfilled}, t)$, and $d(t)$ are the decision variables. The following optimal control model mitigates the disruption impact.

$$\min_{\{r_2(t),\, s(t),\, p(t_{fulfilled}, t),\, d(t)\}} \Delta C. \tag{6}$$

$$\text{Subject to } \dot{b}(t) = r_2(t), \tag{7}$$

$$\dot{I}(t) = r_1 s(t) - d(t), \tag{8}$$

$$d(t) \le 1 - r_2(t), \tag{9}$$

$$d(t) \ge r_1 s(t), \tag{10}$$

$$0 \le p(t_{fulfilled}, t) \le 1 - r_0(t), \tag{11}$$

$$0 \le r_2(t) \le u(t), \tag{12}$$

$$0 \le s(t) \le 1, \tag{13}$$

$$I(0) = I_0,\ b(0) = 0, \tag{14}$$

$$I(T-t_0) = 0. \tag{15}$$

$$r_0(t) = [1 - \theta t_{wait}]^+, \quad \text{where } t_{wait} = T + \frac{b(t)}{P_C - 1} - t. \tag{16}$$

The objective function (6) minimises the disruption cost ΔC incurred during $(0,\ T_{max})$. (7) and (8) capture the dynamics of the backlogged demand and safety inventory during $(0, T)$. (9)-(13) give the maximum and minimum values of the four decision variables. (14) gives the initial quantities of safety stock and backlogged demand. Considering that there is no need to adopt other countermeasures if there is sufficient safety stock, our attention is only on those cases where inventory will be fully depleted before time $T - t_0$ (a point in time before the end of the supply disruption), i.e. (15). (16) presents how the customers' partial back-ordering behaviour without compensation changes in accordance with the backlogged demand under the 'First come first served' principle.

Table 3 shows the strategies derived from the model. For notational simplicity, we omit the time indices t and $t_{fulfilled}$ from this point onwards.

Each optimal joint control decision $\{s^*, r_2{}^*, p^*, d^*\}$ in Table 3 represents a type of strategy. 11 types of strategies are derived as components of constructing the optimal dynamic mitigation strategies for supply disruptions.

I_H and I_L represent two pure strategies of inventory consumption: $\{s^*, r_2{}^*, p^*, d^*\} = \{0, 0, 0, 1\}$, $\{0, r_0{}^*, 0, 1 - r_0{}^*\}$. The manufacturer resumes production through safety inventory. Neither contingent sourcing nor compensation is provided. The difference between these two strategies is the speed of inventory consumption. Under I_H, safety inventory is depleted at the rate of 1 to satisfy all customers, that is, $d^* = 1$ and $-\dot{I}^* = 1$. As a result, no lost sale or backorder appears, b^* remains at $b^*(\tau_0)$. Under I_L, since some customers who are willing to wait without incentives, the manufacturer can consume the inventory at a low rate of $1 - r_0{}^*$ to meet the remaining demand. No lost sale occurs, and b^* increases at the rate of $r_0{}^*$.

SI_H and SI_L represent two mixed strategies that incorporate contingent sourcing and inventory consumption: $\{s^*, r_2{}^*, p^*, d^*\} = \{1, 0, 0, 1\}$, $\{1, r_0{}^*, 0, 1 - r_0{}^*\}$. A contingent sourcing policy is offered to all customers as a priority, and no compensation is given. The difference between these two strategies is about how inventory is consumed. Under SI_H, inventory drops at the rate of $1 - r_1$ to satisfy all those customers who reject the contingent sourcing policy. In other words, no backorder is allowed, i.e. $r_2{}^* = 0$. Under SI_L, inventory is used for satisfying $1 - r_1 - r_0{}^*$ demand. Due to the long lead time and the customers' behaviour, we might have the following situation: $r_0{}^*$ of those customers who refuse the contingent sourcing policy are willing to postpone their orders. That is, $1 - r_1 > r_0{}^*$. Under this situation, the manufacturer allows this fraction of customers to place backorders, then restarts production to meet the remaining demand.

S_H^0, S_H^r and S_L respectively represent three pure sourcing strategies without compensation and inventory consumption: $\{s^*, r_2{}^*, p^*, d^*\} = \{1, 0, 0, r_1\}$, $\{1, r_0{}^*, 0, r_1\}$, and $\{(1 - r_0{}^*)/r_1, r_0{}^*, 0, 1 - r_0{}^*\}$. Under S_H^0 and S_H^r, the sourcing policy is provided to all customers. The difference is that the customers who reject this policy are lost directly in the use of S_H^0, and $r_0{}^*$ of them are backordered automatically in S_H^r. Under S_L, the sourcing policy is provided to $(1 - r_0{}^*)/r_1$ customers.

P_H and P_L represent two pure compensation policies with different levels but without sourcing and inventory consumption: $\{s^*, r_2{}^*, p^*, d^*\} = \{0, 1, 1, 0\}$, $\{0, 1, 1 - r_0{}^*, 0\}$. All customers are motivated to place backorders, i.e. $r_2{}^* = 1$. In view of the customers' behaviour, the compensation is implemented at a lower level in P_L to avoid overcompensation.

SP_H and SP_L represent two mixed strategies that incorporate contingent sourcing and compensation: $\{s^*, r_2{}^*, p^*, d^*\} = \{1, 1 - r_1, 1 - r_1, r_1\}$, $\{1, 1 - r_1, 1 - r_1 - r_0{}^*, r_1\}$. All customers are provided with the contingent sourcing policy, and the ones who refuse this policy are backordered through compensation. Similarly, the compensation level in SP_L is lower than in SP_H.

4. The optimal dynamic mitigation strategies

In this section, we explore the optimal dynamic mitigation strategy. The optimal joint decisions $\{s^*, r_2{}^*, p^*, d^*\}$ with respect to the dynamic optimisation problem (6)-(16) are solved by Pontryagin's Maximum Principle (Seierstad and Sydsæter 1987). First, in view of the inventory dynamics, the problem is decomposed into two stages: before and after the inventory reaches zero. The optimal mitigation strategies and their corresponding time intervals are derived for these two stages. Then, discussing each state transition based on the co-state variables, we present the transition conditions that determine how the optimal strategy changes between the subintervals. Afterward, by combining the optimal strategies for each time interval during the entire disruption period, the optimal dynamic mitigation strategies for hedging against supply failures under diverse circumstances are developed.

By transferring the minimisation problem into a maximisation, the Hamiltonian function $H(r_2, s, d, p, \dot{I}, b, \lambda_1, \lambda_2, t)$ and the Lagrangian function are defined as

$$\begin{aligned} H = &-c_1 d - c_s r_1 s - c_p p - c_l(1 - r_2 - d) - c_h I - (c_0 \\ &+ c_1) r_2 + (c_0 + c_1 + c_h I_0) + \lambda_1 r_2 + \lambda_2 (r_1 s - d). \end{aligned} \tag{17}$$

$$\begin{aligned} L = &H + \mu_1(d - r_1 s) + \mu_2(1 - r_2 - d) + \mu_3 s \\ &+ \mu_4(1 - s) + \mu_5 r_2 + \mu_6(r_0 + p - r_2) + \mu_7 p \\ &+ \mu_8(1 - r_0 - p). \end{aligned} \tag{18}$$

λ_1 and λ_2 represent the co-state variables for b and I. λ_1 can be interpreted as the economic value of the backlogging demand, and λ_2 is the shadow price of the safety inventory. $\mu_1, \ldots, \mu_8$ are the Lagrangian multipliers for the control variables.

Following Pontryagin's Maximum Principle, the optimal joint control $\{s^*, r_2{}^*, p^*, d^*\}$ maximises (17) at every point in time, that is,

$$\begin{aligned} &H(r_2{}^*, s^*, d^*, p^*, \dot{I}^*, b^*, \lambda_1, \lambda_2, t) \\ &\quad = \max_{r_2,\, s,\, d,\, p} H(r_2, s, d, p, \dot{I}, b, \lambda_1, \lambda_2, t). \end{aligned} \tag{19}$$

The following necessary conditions are required.

$$\partial L/\partial r_2 = c_l - (c_0 + c_1) + \lambda_1 - \mu_2 + \mu_5 - \mu_6 = 0. \tag{20}$$

$$\partial L/\partial p = -c_p + \mu_6 + \mu_7 - \mu_8 = 0. \tag{21}$$

Table 4. The optimal strategies in Inventory Stage 1.

			The conditions for optimal strategy	
r_0	The optimal strategy	The co-state λ_1	$\lambda_2 < c_l - c_1$	$\lambda_1 + \lambda_2 < c_0 + c_p$
0	I_H	constant	$\lambda_2 < c_s$	
+	I_H	constant		$\lambda_1 + \lambda_2 < c_0$
+	I_L	F_1		$\lambda_1 + \lambda_2 > c_0$
0	SI_H	constant	$\lambda_2 > c_s$	
+	SI_H	constant		$\lambda_1 + \lambda_2 < c_0$
$0 < r_0 < 1 - r_1$	SI_L	F_1		$\lambda_1 + \lambda_2 > c_0$

Table 5. The optimal strategy in Inventory Stage 2.

			The conditions for each strategy	
r_0	The optimal strategy	The co-state λ_1	λ_1	λ_2
0	S_H^0	constant	$\lambda_1 < M_1 + c_p$	$> c_l - c_1$
	SP_H	constant	$c_p + M_1 < \lambda_1 < c_p + M_2$	$> c_s$
	P_H	constant	$\lambda_1 > c_p + M_2$	
$0 < r_0 < 1 - r_1$	S_H^r	F_2	$M_1 < \lambda_1 < c_p + M_1$	$> c_l - c_1$
	SP_L	F_4	$c_p + M_1 < \lambda_1 < c_p + M_2$	$> c_s$
	P_L	F_4	$\lambda_1 > c_p + M_2$	
$r_0 \geq 1 - r_1$	S_L	F_3	$M_2 < \lambda_1 < c_p + M_2$	$> c_s$
	P_L	F_4	$\lambda_1 > c_p + M_2$	

$$\partial L/\partial d = -c_1 + c_l - \lambda_2 + \mu_1 - \mu_2 = 0. \tag{22}$$

$$\partial L/\partial s = -c_s r_1 + \lambda_2 r_1 - \mu_1 + \mu_3 - \mu_4 = 0. \tag{23}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial L}{\partial b} = \frac{-\theta}{P_C - 1}(\mu_6 - \mu_8), \text{ if } r_0 > 0,$$
$$\frac{d\lambda_1}{dt} = -\frac{\partial L}{\partial b} = 0, \text{ if } r_0 = 0. \tag{24}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial L}{\partial I} = c_h. \tag{25}$$

Combining (21) with (24), λ_1 is defined as $\frac{d\lambda_1}{dt} = -\frac{\partial L}{\partial b} = -\frac{\theta}{P_C-1}(c_p - \mu_7)$ if $r_0 > 0$, and $\lambda_1 =$ *constant* if $r_0 = 0$.

Due to the un-constrained terminal demand state $b(T - t_0)$, the transversality condition (Seierstad and Sydsæter 1987) is further required for λ_1.

$$\lambda_1(T - t_0) = 0. \tag{26}$$

The co-state variable λ_1 remains constant or drops over time (the specific expressions are given in Tables 4 and 5), and reaches 0 at the end. When $r_0 > 0$, a number of customers automatically place backorders. An additional backorder requires a recovery time $\frac{1}{P_C-1}$, thus the value of holding backorders decreases over time. In particular, if backorders are facilitated through compensation (here, $\mu_7 = 0$ and $p > 0$ in (31)), the slope of λ_1 equals $-\frac{\theta}{P_C-1}c_p$. The co-state variable λ_2 is an increasing function of t, and equals the inventory holding cost c_h. Then,

$$\lambda_2 = \lambda_2(0) + c_h t. \tag{27}$$

The Lagrangian multipliers $\mu_1, \ldots, \mu_8$ must satisfy the complementary slackness conditions (28)-(31).

$$\mu_1 \geq 0, \text{ and } \mu_1(d - r_1 s) = 0;$$
$$\mu_2 \geq 0, \text{ and } \mu_2(1 - r_2 - d) = 0. \tag{28}$$
$$\mu_3 \geq 0, \text{ and } \mu_3 s = 0;$$
$$\mu_4 \geq 0, \text{ and } \mu_4(1 - s) = 0. \tag{29}$$
$$\mu_5 \geq 0, \text{ and } \mu_5 r_2 = 0;$$
$$\mu_6 \geq 0, \text{ and } \mu_6(r_0 + p - r_2) = 0. \tag{30}$$
$$\mu_7 \geq 0, \text{ and } \mu_7 p = 0;$$
$$\mu_8 \geq 0, \text{ and } \mu_8(1 - r_0 - p) = 0. \tag{31}$$

Next, we analyse the optimal joint controls $\{s^*, r_2{}^*, p^*, d^*\}$ in (16), along with the necessary conditions that determine their corresponding time intervals.

Due to the linearity of (17) in all controls, the optimal joint controls $\{s^*, r_2{}^*, p^*, d^*\}$ can only occur at the boundaries, that is,

$$s^* \in \{0, 1, d/r_1\}; \quad r_2{}^* \in \{0, r_0 + p\};$$
$$p^* \in \{0, 1 - r_0, r_2 - r_0\}; \quad d^* \in \{1 - r_2, r_1 s\}. \tag{32}$$

(32) can be interpreted as follows. The optimal sourcing s^* can obtain three values: the minimum 0, the maximum 1, and an undetermined medium value that is obtained on the boundary of $s = d/r_1$. The manufacturer seeks no sourcing if $s^* = 0$, and provides all customers with the sourcing strategy in priority if $s^* = 1$. The solution $s^* = d/r_1$ indicates that the production during the interruption is resumed by contingent sourcing only, without

using inventory. For the optimal backorder rate $r_2{}^*$, we have two cases. If $r_2{}^* = 0$, no backorder occurs during the disruption. If $r_2{}^* = r_0 + p$, the backorder rate is increased from r_0 to $r_0 + p$ by providing a compensation p, where the optimal compensation level p^* could be the minimum 0, the maximum $1 - r_0$, or an undetermined medium value on the boundary of $p = r_2 - r_0$. In addition to the solution that occurs on the boundary of $d = r_1 s$, the optimal satisfied demand can also be $d^* = 1 - r_2$, which means that all unfilled demands are backordered. Note that r_0 is the state-dependent piecewise function given in (16).

Based on the two states of r_0 and the multiple possible optimal values of each decision variable, there can be a large number of joint controls. Through an analysis of the necessary conditions for these joint controls, the optimal joint controls can be achieved. Next, we further specify the undetermined potential optimal values introduced in (32) and accordingly construct the optimal joint controls. To this end, we address the analysis from two stages.

Due to the monotonicity of the inventory I, there is a single transition time point, denoted as t_{i0} where $I = 0$ is activated. Accordingly, the optimisation problem involves the following two stages.

(i) Inventory Stage 1:$[0, t_{i0})$

The system maintains a positive amount of safety stock. As a result of inventory consumption, the satisfied demand has to be $d > r_1 s$. Based on (32), the optimal satisfied demand is determined as $d^* = 1 - r_2$.

(ii) Inventory Stage 2: $(t_{i0}, T - t_0]$

The system carries an empty inventory. Without an available inventory, demand can only be fulfilled through contingent sourcing, that is $d^* = r_1 s$.

4.1. The optimal strategy in inventory stage 1

Given $d^* = 1 - r_2$, the optimal values that need to be further specified from (32) are $s^*, r_2{}^*, p^*$. In order to do this, we substitute $d^* = 1 - r_2$ into (17), and derive the Hamiltonian function as

$$H_1 = (\lambda_1 + \lambda_2 - c_0) r_2 - c_p p + (\lambda_2 - c_s) r_1 s - \lambda_2 - c_h I + c_0 + c_h I_0. \tag{33}$$

By maximising (33), the optimal values of $s^*, r_2{}^*, p^*$ are achieved. Then, after analysing the necessary conditions (20)-(24) for the optimal controls $\{s^*, r_2{}^*, p^*, d^* = 1 - r_2\}$, the optimal decisions in Inventory Stage 1 are shown in Table 4.

As indicated in Table 4, based on the states of the customers' behaviour (Demand State 1 where $r_0 = 0$ and Demand State 2 where $r_0 > 0$), the optimal decision in Inventory Stage 1 can be one of the following four strategies: I_H, I_L, SI_H, and SI_L (see the explanation in Table 3). Inventory is consumed at different speeds during the periods (τ_{ki}, τ_{ko}), which is when these strategies are performed, and reaches zero at time t_{i0}. Here, $k = 1, \ldots, 4$. Therefore, t_{i0} is identified as $t_{i0} = \max\{\tau_{1o}, \tau_{2o}, \tau_{3o}, \tau_{4o}\}$.

$$I_0 = \int_{\tau_{1i}}^{\tau_{1o}} 1 dt + \int_{\tau_{2i}}^{\tau_{2o}} (1 - r_0{}^*) dt + \int_{\tau_{3i}}^{\tau_{3o}} (1 - r_1) dt + \int_{\tau_{4i}}^{\tau_{4o}} (1 - r_0{}^* - r_1) dt. \tag{34}$$

The conditions defining the time intervals are tightly linked to the cost factors: c_0, $c_0 + c_p$, c_s, c_l and the co-state variables λ_1 and λ_2. Regarding λ_2 as the shadow price of a unit inventory, and λ_1 as the economic value of an additional backorder, we see: In the process of resuming production to satisfy a unit demand, a total cost of $\lambda_2 + c_1$ is incurred if raw materials are taken from the inventory, and a cost of $c_1 + c_s$ is incurred if contingent replenishments are procured. On the other hand, if the manufacturer, rather than meeting the demand during the interruption, leaves it as a backorder, then the amount of $c_0 + c_1$ will be required to fulfil the order during the recovery period. Furthermore, if the backorder is generated from compensation, an extra cost c_p is generated on top of $c_0 + c_1$. Thus, the costs of a backorder with and without compensation are $c_0 + c_1 - \lambda_1$ and $c_0 + c_1 + c_p - \lambda_1$, respectively.

The above conditions actually indicate the superiority of the following decisions: satisfying demand with contingent souring or inventory consumption, backordering with or without compensation, and allowing lost demand.

(i) $\boldsymbol{\lambda_2 < c_l - c_1}$

The condition guarantees that the consumption of inventory is superior to doing nothing and incurring lost sales. Therefore, no lost sale appears in Inventory Stage 1.

(ii) $\boldsymbol{\lambda_1 + \lambda_2 < c_0 + c_p}$

That is, $\lambda_2 + c_1 < c_0 + c_1 + c_p - \lambda_1$, the consumption of inventory to satisfy demand is superior to compensating customers for their waiting time. Thus, compensation is excluded under this condition.

(iii) $\boldsymbol{\lambda_2 < c_s}$

Consuming inventory is cheaper than procuring emergency replenishments. Thus, contingent sourcing is excluded if this condition is met. On the contrary, emergency procurement should be employed if $\lambda_2 > c_s$. Note that, this condition might be opposite to what common sense suggests. The reason is as follows. Our study focuses on the case that safety inventory is not sufficient for dealing with outages. Therefore, the shadow price of consuming inventory is not only related to the inventory holding cost, but also linked to the loss incurred after the inventory is entirely depleted.

(iv) $\lambda_1 + \lambda_2 < c_0$

That is, $\lambda_2 + c_1 < c_0 + c_1 - \lambda_1$, the consumption of inventory is superior to backordering without compensations. No backorder is allowed under this condition. The condition answers an important question for the manufacturer: whether or not accept backorders in Inventory Stage 1 while some customers in the market are willing to postpone their purchases without any incentives (i.e. $r_0 > 0$)? The results suggest that the manufacturer allows a positive backorder rate, that is, implementing the strategies I_L and SI_L if $\lambda_1 + \lambda_2 > c_0$ and $r_0 > 0$. In particular, if $r_0 = 0$, no customer is willing to backorder automatically, and there is no need to discuss this question.

Based on the proposed strategies and their corresponding conditions in Table 4, we construct the optimal dynamic strategy from the beginning of disruption.

At the occurrence of disruption, no backorder has accumulated in the production system, thus, customers who chose to postpone their orders will be satisfied immediately after supply resumes at time T. That is, $t_{wait}(0) = T$, and $r_0(0) = [1 - \theta T]^+$ customers are willing to backorder without compensation. With the initial state of the customers' behaviour, we shed light on the initial strategy for alleviating disruption.

Proposition 1: ***(the initial strategy)*** *The optimal strategy starts at $t = 0$ with*

(i) *I_H or SI_H, if (a) $\theta T \geq 1$ or (b) $\theta T < 1$ and $\lambda_1(0) + \lambda_2(0) < c_0$;*

(ii) *I_L or SI_L, if $\theta T < 1$ and $\lambda_1(0) + \lambda_2(0) > c_0$.*

It is preferable for the manufacturer to satisfy all demand at the occurrence of disruption through pure inventory consumptionI_H or the combination policy SI_H, if we have one of the following two scenarios. (a) $\theta T \geq 1$: The disruption is too long or customers are sensitive so that no one is willing to wait without any compensation. (b) $\theta T < 1$ and $\lambda_1(0) + \lambda_2(0) < c_0$: some customers are willing to backorder without compensation. Nonetheless, the economic value of placing a backorder is less than the cost of consuming inventory. Therefore, no backorder or lost sale should occur. Conversely, if $\theta T < 1$ and $\lambda_1(0) + \lambda_2(0) > c_0$, partial backordering is more advantageous.

As disruption continues, the inventory and the backlogged demand dynamically change, along with the value of backlogging a demand and consuming a unit of inventory. Consequently, the initial strategy may no longer be advantageous. As stated in Table 4, while inventory remains positive in Inventory Stage 1, there are two types of time points when transitions between strategies might occur: when the state of backlogged demand

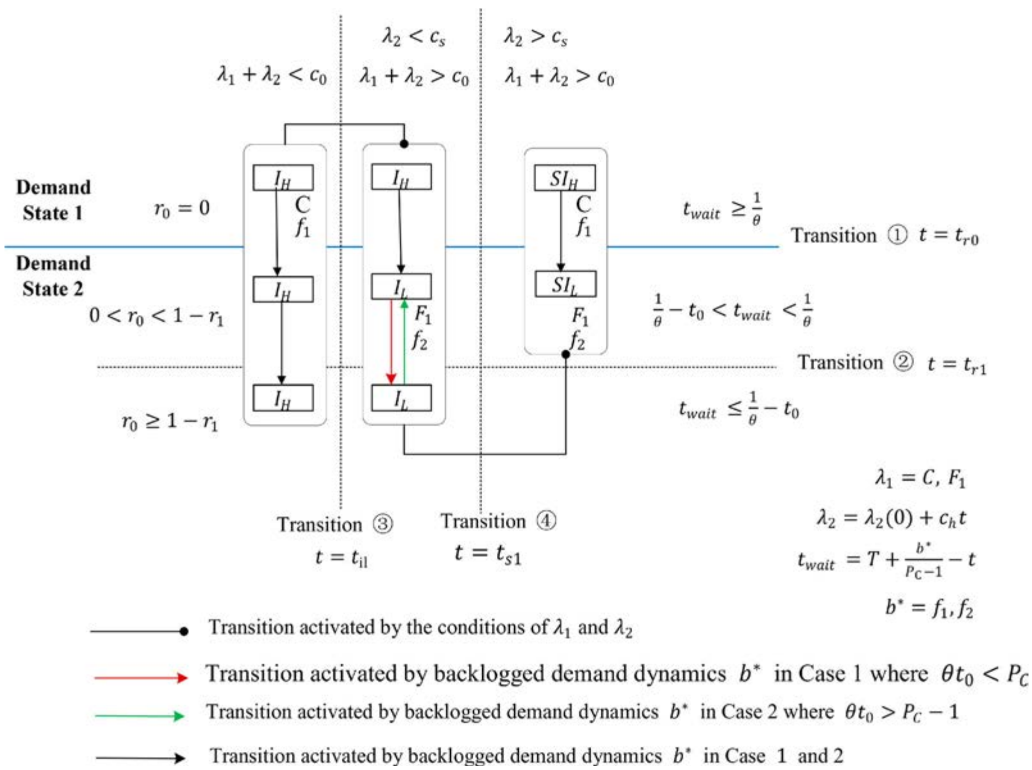


Figure 2. Possible transitions between strategies in Inventory Stage 1.

changes or when the conditions defined by the co-state variables λ_1 and λ_2 of backlogged demand and inventory are no longer fulfilled. By analysing the corresponding dynamics of r_0, λ_1 and λ_2 under the implementation of each strategy, we determine when and how to switch the optimal strategies in Inventory Stage 1, as presented in Figure 2 (the analysis is detailed in the online supplement).

As depicted in Figure 2, transitions between strategies in Inventory Stage 1 are as follows:

(i) In Demand States 1–2: the optimal strategy changes from I_H and I_L into SI_H and SI_L at time t_{s1} when $\lambda_2 = c_s$ is activated (transition ④). In other words, regardless of the state of customers' behaviour, contingent sourcing should be employed on top of inventory consumption at the critical time when the shadow price of inventory exceeds the contingent sourcing cost c_s.

(ii) In Demand State 2 where $r_0 > 0$: The optimal strategy changes from I_H into I_L at time t_{il} when $\lambda_1 + \lambda_2 = c_0$ is activated (transition ③).

(iii) At the critical time when the demand state changes from 1 into 2 (transition ①): the optimal strategy changes from I_H and SI_H into I_L and SI_L during the period where $\lambda_1 + \lambda_2 > c_0$.

The following managerial insights are observed from Figure 2.

Proposition 2: *When the system holds inventory,*

(i) *it is never optimal to provide compensation for backorders;*

(ii) *contingent sourcing is employed on top of inventory consumption in the time interval where $\lambda_2 > c_s$. In particular, the contingent sourcing is implemented from the outset if $\lambda_2(0) > c_s$.*

(iii) *It is optimal not to backorder unless $\lambda_1 + \lambda_2 > c_0$ and $r_0 > 0$.*

It is worth noting that when the manufacturer still carries a positive inventory, it is profitable to backlog demands if the customers are willing to place backorders without any compensation, and the value of backlogging demand is superior over meeting demand via inventory.

4.2. The optimal strategy in inventory stage 2

Like Section 4.1, we start the analysis with specifying the optimal values of $s^*, r_2{}^*, p^*$. According to (32), the optimal satisfied demand $d^* = r_1 s$ includes two scenarios: (a) $d^* = r_1 s = 1 - r_2$; (b) $d^* = r_1 s < 1 - r_2$.

(a) Substituting $d^* = r_1 s$ into (17), the Hamiltonian function is derived as

$$H_2 = [c_l - (c_0 + c_1) + \lambda_1] r_2 - c_p p + (-c_1 - c_s + c_l) \times r_1 s - c_l + (c_0 + c_1 + c_h I_0). \tag{35}$$

(b) Substituting $r_1 s = 1 - r_2$ into (32),

$$H_3 = (\lambda_1 + c_s - c_0) r_2 - c_p p - c_s + c_0 + c_h I_0. \tag{36}$$

By maximising (35) and (36), the optimal values of $s^*, r_2{}^*, p^*$ are obtained in the above two scenarios. Then, by analysing the necessary conditions (20)-(24) for $\{s^*, r_2{}^*, p^*, d^* = r_1 s = 1 - r_2\}$ and $\{s^*, r_2{}^*, p^*, d^* = r_1 s < 1 - r_2\}$, the optimal joint decisions in Inventory Stage 2 are determined. The results include six types of optimal strategies and the corresponding conditions, as given in Table 5.

In view of the customers' behaviour (Demand State 1 where $r_0 = 0$ and Demand State 2 where $r_0 > 0$), the optimal decision manufacturer should make in Inventory Stage 2 falls into three forms of pure sourcing S_H^0, S_H^r, S_L, two forms of pure compensation P_H, P_L, and two forms of mixed strategies SP_H, and SP_L (see the explanation in Table 3). The conditions mainly depend on the cost factors and the co-state variable λ_1, which determine the priority among the following four decisions: satisfying demand with contingent souring, backordering with or without compensation, and letting demand be lost sales.

(i) $\lambda_1 > c_p + c_0 - c_s$

Together with $c_s + c_1 < c_l$, we see that $c_p + c_0 - \lambda_1 < c_s < c_l - c_1$. Backordering with compensation has more advantages than other choices. Therefore, the pure compensation policies P_H and P_L are implemented, and no lost sale occurs. In particular, if $r_0 > 0$, some customers are willing to backorder without compensation, thus P_L should be employed to avoid overcompensation.

(ii) $c_p + c_0 + c_1 - c_l < \lambda_1 < c_p + c_0 - c_s$ and $r_0 < 1 - r_1$

Backordering with compensation is superior to incurring lost sales, but inferior to sourcing. Thus, sourcing is the first option offered to all customers. Then, for the remaining customers $1 - r_1$, compensation is provided to motivate growth in the backorder rate from r_0 to $1 - r_1$. That is, the strategies SP_H and SP_L are applied.

(iii) $\lambda_1 > c_p + c_0 - c_s$ and $r_0 \geq 1 - r_1$

If $1 - r_1 = \theta t_0 \leq r_0$, the lead time is not too long, and no customer is willing to automatically postpone his/her orders after having refused to wait for delayed delivery from contingent sourcing. Under this situation, the combination of sourcing and compensation will never happen. Therefore, pure sourcing S_L is employed since $\lambda_1 > c_p + c_0 + c_1 - c_l$.

(iv) $\boldsymbol{\lambda_1 > c_p + c_0 + c_1 - c_l}$

That is, $c_p + c_0 - \lambda_1 > c_l - c_1 > c_s$. Satisfying demand with contingent souring has more advantages than other choices. Therefore, pure sourcing policies S_H^0 and S_H^r are implemented.

Similar to Figure 2, Figure 3 is also presented to explain how the optimal strategies switch in Inventory Stage 2 (the analysis is detailed in the online supplement). Together with the terminal value $\lambda_1(T - t_0) = 0$, the optimal terminating strategy is determined as shown in Proposition 3.

Proposition 3: ***(the terminating strategy)*** *At the end of available contingent sourcing, the optimal strategy terminates with*

(i) *pure compensation, if $c_p + c_0 < c_s$;*
(ii) *a combination of compensation and sourcing, if $c_s < c_p + c_0 < c_l - c_1$ and $t_0 > \frac{r_0(T-t_0)}{\theta}$;*
(iii) *pure sourcing if (a) $c_s < c_p + c_0 < c_l - c_1$ and $t_0 < \frac{r_0(T-t_0)}{\theta}$; (b) $c_p + c_0 > c_l - c_1 > c_s$.*

Where, $r_0(T - t_0) = \left[1 - \theta \frac{b(T-t_0)}{P_C - 1}\right]^+$.

As illustrated in Proposition 3, the cost factors hugely determine the optimal terminal strategy. Pure compensation is activated at the termination time if the compensation cost is small. Conversely, pure 'Sourcing' is preferable at the end of available contingent sourcing if the cost and the lead time t_0 offer a significant advantage. Otherwise, the combination of compensation and sourcing is realised.

4.3. The optimal transitions between strategies at time t_{i0}

In order to construct the optimal dynamic strategies for the entire disruption period, we need to combine the strategies of Inventory Stages 1 and 2. That is, it is important to address the following discussion: Is it possible for the optimal dynamic strategy to switch from I_H, I_L, SI_H, and SI_L to S_H^0, S_H^r, S_L, SP_H, SP_L, P_H, and P_L at the critical time t_{i0} where inventory reaches zero?

Note that, since the inventory state changes at time t_{i0}, a jump of the co-state variable λ_2 might occur. The following jump condition (Seierstad and Sydsæter 1987) holds:

$$\begin{aligned} &H(r_2{}^*(t_{i0}{}^-), s^*(t_{i0}{}^-), p^*(t_{i0}{}^-), d^*(t_{i0}{}^-), b^*(t_{i0}{}^-),\\ &\quad I^*(t_{i0}{}^-), \lambda_1(t_{i0}{}^-), \lambda_2(t_{i0}{}^-), t_{i0})\\ &= H(r_2{}^*(t_{i0}{}^+), s^*(t_{i0}{}^+), p^*(t_{i0}{}^+), d^*(t_{i0}{}^+),\\ &\quad b^*(t_{i0}{}^+), I^*(t_{i0}{}^+), \lambda_1(t_{i0}{}^+), \lambda_2(t_{i0}{}^+), t_{i0}). \qquad (37) \end{aligned}$$

Table 6. The condition of transitions at time t_{i0}.

Possible transitions between strategies		
From	To	Condition for each transition
I_H	SP_H	$\lambda_2{}^- = c_s$ and $\lambda_1{}^+ = -c_s + c_0 + c_p$
	P_H	$\lambda_2{}^- = c_p + c_0 - \lambda_1{}^+$
I_L	P_L	$\lambda_2{}^- = c_p + c_0 - \lambda_1{}^+$
	SP_L	$\lambda_2{}^- = c_s$ and $r_0(t_{i0}) = 1 - r_1$
	S_L	$\lambda_2{}^- = c_s$
	S_H^r	$\lambda_2{}^- = c_s$ and $r_0(t_{i0}) = 1 - r_1$
SI_H	S_H^0	$\lambda_2{}^- = c_l - c_1$
	SP_H	$\lambda_2{}^- = -\lambda_1{}^+ + c_0 + c_p$
SI_L	S_H^r	$\lambda_2{}^- = c_l - c_1$
	SP_L	$\lambda_2{}^- = c_p + c_0 - \lambda_1{}^+$

Where, $\lambda_2{}^- = \lambda_2(t_{io}{}^-)$ and $\lambda_1{}^+ = \lambda_1(t_{io}{}^+)$.

In view of the continuity property of the demand state variable b, together with the jump condition (37), we compare the terminating conditions of Inventory Stage 1 with the entering conditions of Inventory Stage 2, and exclude the impossible candidates for entering into Inventory Stage 2 in Figure 4. The condition for each possible transition is given in Table 6.

As indicated in Figure 4 and Table 6, each transition is determined by the critical economic value of facilitating an additional backorder and consuming inventory at the critical time t_{i0}.

4.4. The optimal dynamic strategies during disruption

After establishing the optimal dynamic strategies in Inventory Stages 1 and 2 according to the customers' behaviours (Demand States 1 and 2), and the transitions between strategies at the time when the inventory is fully depleted, we comprehensively present the optimal dynamic mitigation strategies for the entire period $(0, T - t_0)$, as shown in Figure 5.

As depicted in Figure 5, the optimal dynamic strategies fall into two cases, dependent on the customers' sensitivity θ to the waiting time, the lead time t_0 of contingent sourcing, and the manufacturer's production capacity P_C after the end of disruption: **Case 1** where $\theta t_0 < P_C - 1$ and **Case 2** where $\theta t_0 > P_C - 1$. In each case, three portfolios of optimal dynamic strategies are established dependent on the cost factors.

In particular, no contingent sourcing should be employed at any point in time during disruption, if

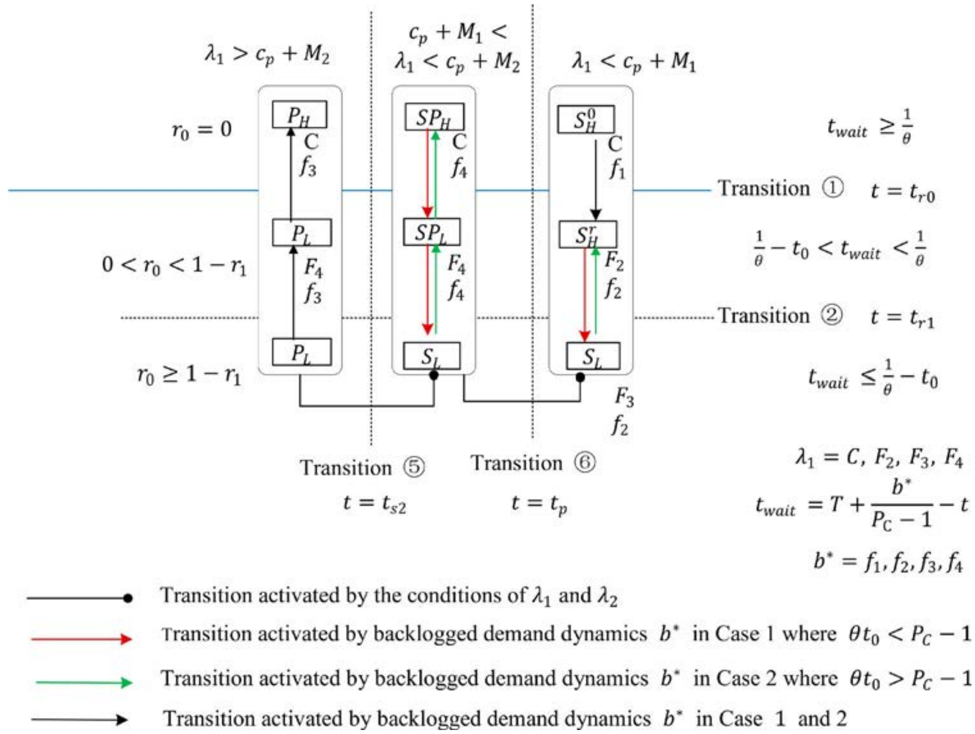


Figure 3. Possible transitions between strategies in Inventory Stage 2.

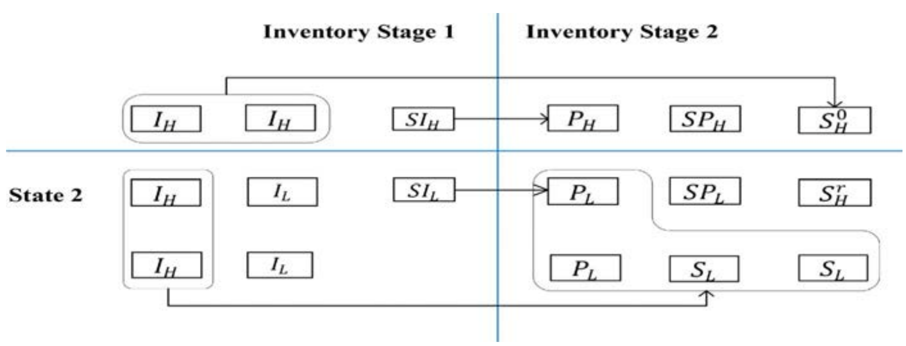
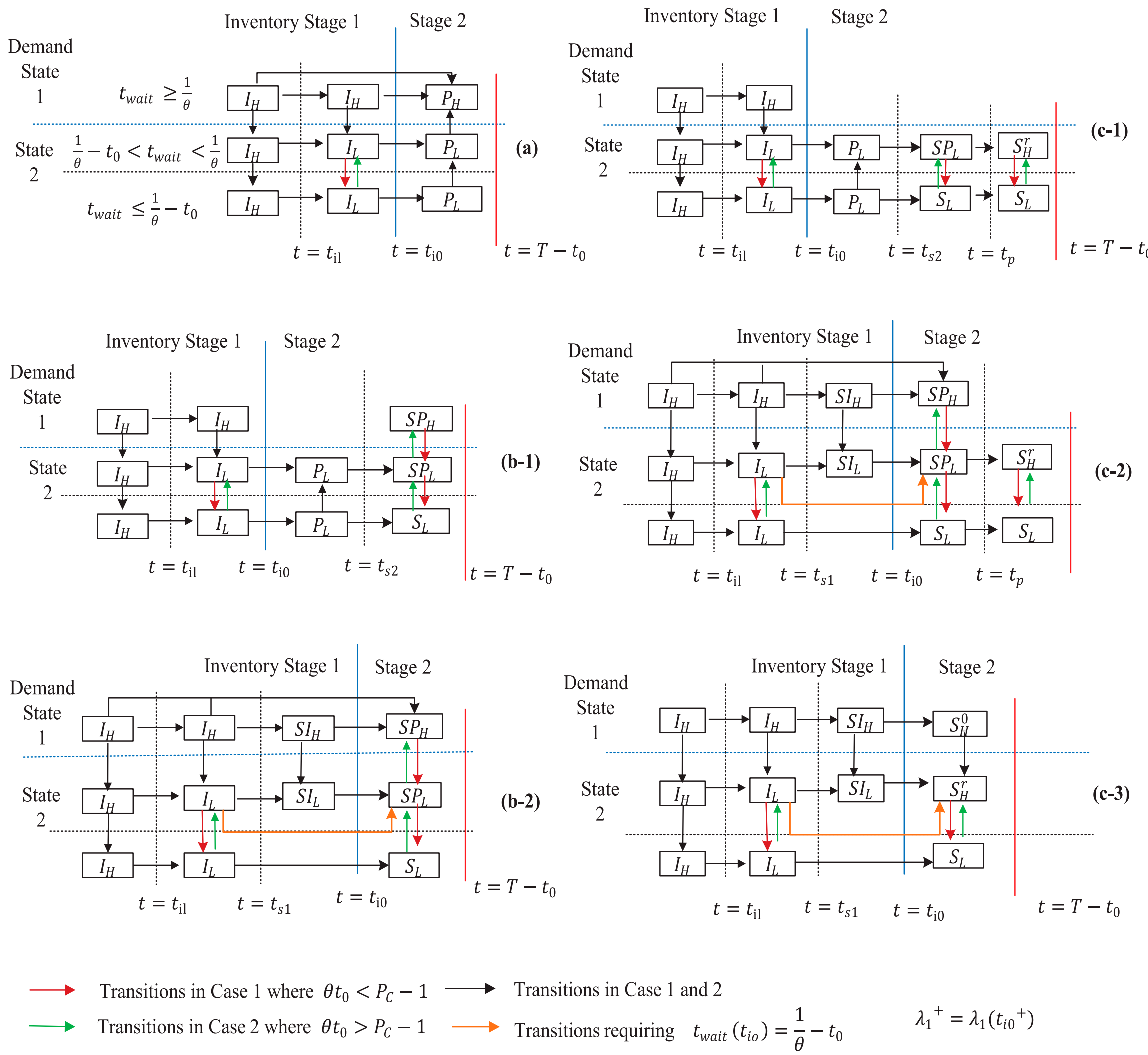


Figure 4. Impossible transitions between strategies at time t_{i0}.

the compensation cost exhibits a significant advantage, i.e. $c_p + c_0 < c_s$. Conversely, if $c_p + c_0 > c_l - c_1$ and $\lambda_1(t_{i0}{}^+) < c_p + c_0 - c_l + c_1$, compensation should be excluded from the entire period. Otherwise, a dynamic mixed strategy that jointly adjusts the time and quantity (level) of the following decisions is realised: backlogging demand, contingent sourcing, and customer compensation.

Each optimal dynamic strategy is represented by a path from the initial time 0 to the terminal time $T - t_0$(see Figure 5), as constructed by 11 types of strategy components: I_H, I_L, SI_H, SI_L, S_H^0, S_H^r, S_L, SP_H, SP_L, P_H, and P_L. The strategy components are given in closed forms and explained in Table 3. They provide specific guidance for the manufacturer on the real-time joint decision. The conditions under which the paths are optimal to manage supply disruptions are presented in closed forms. They consist of the initial condition critically linked to the disruption length (Proposition 1), the transition condition defined by the state dynamics and co-state variable of backlogged demand and inventory (Figures 2–4), and the terminating condition given by the cost factors (Proposition 3).

As we observe the possible paths based on the initial state of the customers' behaviour, i.e. $t_{wait}(0) = T$, an important insight is also revealed. For short disruptions, the optimal dynamic strategies are mainly composed of pure strategies such as I_H, I_L, S_H^0, S_H^r, S_L, P_H, and P_L. As opposed to this, as for long disruptions, the combined strategies that simultaneously incorporate two countermeasures (SP_H, SP_L, SI_H, SI_L) are required during some periods. In particular, by controlling the customers'



(a) $c_s > c_p + c_0$

(b) $c_s < c_p + c_0 < c_l - c_1$: $\mathbf{(b-1)}$ $\lambda_1^+ > c_p + c_0 - c_s$; $\mathbf{(b-2)}$ $\lambda_1^+ \le c_p + c_0 - c_s$

(c) $c_p + c_0 > c_l - c_1$: $\mathbf{(c-1)}$ $\lambda_1^+ > c_p + c_0 - c_s$; $\mathbf{(c-2)}$ $c_1 + c_p + c_0 - c_l < \lambda_1^+ \le c_p + c_0 - c_s$; $\mathbf{(c-3)}$ $\lambda_1^+ \le c_1 + c_p + c_0 - c_l$

Figure 5. The optimal dynamic strategies during disruption.

behaviour at the state of $r_0 = 0$, Corollary 1 presents the optimal dynamic strategies for hedging against relatively long disruptions.

Corollary 1: (i) *It is preferable to employ the dynamic strategy $I_H - SI_H - SP_H$, if $c_s < c_p + c_0 < c_l - c_1$, $t_0 < \min\{\frac{c_h I_0}{(c_0+c_p-c_s)\theta}, \frac{1}{\theta}\}$, and*

$$T \ge \max\left\{\frac{P_C - 1}{\theta^2 t_0} + t_{i0} + t_0, \frac{1}{\theta} + t_{i0} + t_0\right\}, \quad where$$

$$t_{i0} = I_0 + (1 - \theta t_0)\frac{c_0 + c_p - c_s}{c_h}. \quad (38)$$

(ii) *It is preferable to employ $SI_H - SP_H$, if $c_s < c_p + c_0 < c_l - c_1$, $\frac{c_h I_0}{(c_0+c_p-c_s)\theta} < t_0 < \frac{1}{\theta}$, and*

$$T \ge \max\left\{\frac{P_C - 1}{\theta^2 t_0} + \frac{I_0}{\theta t_0} + t_0, \frac{1}{\theta} + \frac{I_0}{\theta t_0} + t_0\right\}. \quad (39)$$

(iii) *It is preferable to employ $I_H - P_H$, if $c_p + c_0 < c_s$, and*

$$T \ge \frac{1}{\theta} + I_0 + t_0. \quad (40)$$

If the disruption lasts extremely long that no customer is willing to backorder at any time without compensation, priority is given to the following countermeasures. (a) If $c_s > c_p + c_0$, production is resumed to meet all demand in Inventory Stage 1, then all customers are immediately compensated at the maximum level in Inventory

Stage 2. (b) If $c_s < c_p + c_0 < c_l - c_1$, contingent sourcing is utilised while inventory is consumed or customers are compensated. In particular, if the lead time is long, the contingent sourcing policy should be provided immediately at the beginning of the disruption to overcome this shortfall. Observing (38)-(40)*, we also find that the advisability of utilising $SI_H - SP_H$ grows with θ and c_p, and drops with c_s. In other words, the manufacturer should reroute to secondary replenishments while consuming inventory from time $t = 0$ even if the disruption will not last extremely long or the lead time is short due to one of the following events happens: customers become impatient; the sourcing cost is small; the compensation cost is large.*

4.5. Numerical analysis

In order to illustrate the approach and value of different strategies, we conduct a numerical analysis to visually present the optimal dynamic mixed strategies for one sub-scenario and generate further insights into the roles of parameters, such as the lead time t_0 and the cost $c_s$of contingent sourcing, the compensation cost c_p, and the disruption length T.

As indicated in Figure 5, neither contingent sourcing nor customer compensation should be employed during disruption if the compensation or sourcing cost is extremely small. Therefore, to present a more general illustration, we consider the following sub-case (Case 2(b-2) in Figure 5): $c_s < c_p + c_0 < c_l - c_1$, $\lambda_1{}^{+}(t_{i0}) > -c_s + c_0 + c_p$, and $\theta t_0 > P_C - 1$. That means, both compensation and contingent sourcing have certain advantages in terms of cost, but the lead time of contingent sourcing is relatively long (this sub-case is common when contingent sourcing is offshore). To this end, we establish a basic set as follows: $P_C = 1.5, c_0 = 10, c_l = 100, c_1 = 10, c_h = 0.2$, $\theta = 0.01$, $I_0 = 100$. Our model is continuously time-dependent and the demand rate is normalised as '1'. By assuming that a time unit is one hour, the basic setting stands for the following circumstance: a manufacturer produces and sells 24 units of products per day, and carries around 4.2 days of safety stocks in raw materials, while customers in the market will renounce their willingness to wait without any incentive countermeasures in 4.2 days. Note that all the strategies are presented in closed form, and the following main findings still hold under other settings. The numerical analysis based on this specific setting is mainly to visually illustrate how the approach and strategies proposed in this study can be utilised in practice.

According to Figure 5 (Case 2(b-2)), the optimal dynamic strategies under this setting could be in multiple paths constructed by $I_H, I_L, SI_H, SI_L, SP_H, SP_L$. Let the compensation cost, the cost and lead time of contingent sourcing, and the disruption length vary. We first observe how the structure of optimal dynamic strategies will change according to these factors, as shown in Figure 6.

In Figure 6, the blank part represents the other paths given in Case 2 (b-1). The results show that when the contingent sourcing cost is relatively small (Case 2 (b-2)), the optimal strategies mainly are of three forms: the dynamic strategies $SI_H - SP_H$ and $I_H - SI_H - SP_H$ for long supply failures, and $I_H - SI_H - SI_L - SP_L$ for relatively short disruptions with a high compensation cost. In other words, it is optimal to implement contingent sourcing from time t_{s1} before the inventory is used up, or from the beginning of the disruption. As the lead time or the sourcing cost increases, the advisability of utilising such strategies decreases. As a further illustration, Figure 7 presents the specific joint decisions of $I_H - SI_H - SI_L - SP_L$ under $c_s = 15$, $t_0 = 65$, and $c_p = 45$, and the variation trends for different disruption lengths.

In Figure 7(a), the x-axis is the time after the appearance of supply failure. The y-axis presents the optimal dynamic joint decision: the inventory dynamics I^* from I_0 =100, and the zoomed lines (r_2^*, s^*, p^*) in (0,1). The corresponding dynamics of the backlogged demand and customer behaviour are also presented. The result provides the following suggestions for the manufacturer. In the first period (0, t_{r0}) of disruption, no customer is willing to backorder without compensation due to the long waiting time. In order to avoid lost sales, the manufacturer adopts the strategy $I_H - SI_H$: resuming production through the inventory to meet the total demand '1' at the first phase (0, t_{s1}), then utilising contingent sourcing in conjunction with the inventory at the second phase (t_{s1}, t_{r0}). By doing so, no lost sale or backlogged demand is incurred. As a result, the customers who arrive subsequently in the second period will experience a shorter waiting time, and r_0^* of them are willing to postpone purchases. Hence, during the second period (t_{r0}, t_{i0}), the manufacturer should accept backorders and satisfy the customers who are not willing to wait, i.e. he should utilise the strategy SI_L. After the safety inventory is used up at time t_{i0}, the strategy SP_L is adopted. That is, we first provide the contingent sourcing policy to all customers. Then, for the ones who refuse this policy, we provide a level of $1 - r_1 - r_0^*$ compensation to have them backorder.

Figure 7(b) shows how these decisions change with different disruption lengths. Note that, in the process of using $I_L - SI_H - SI_L - SP_H$, the contingent sourcing is always provided to all customers as a priority after time t_{s1}. Therefore, we focus on the variation trend of t_{s1} with $T - t_0$. As the disruption length T increases, the results point towards two suggestions: the speed of consuming

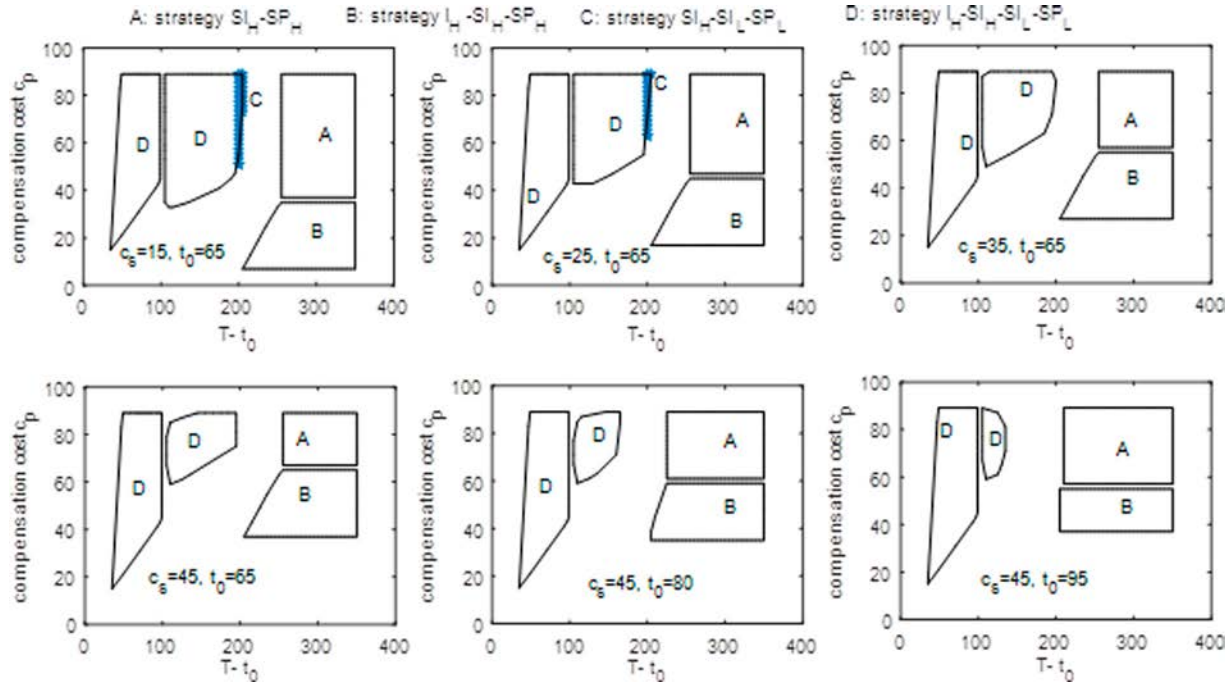


Figure 6. The optimal dynamic strategies under different c_p, c_s, T and t_0.

inventory should be reduced; the backorders should be allowed later and the contingent sourcing should be implemented earlier.

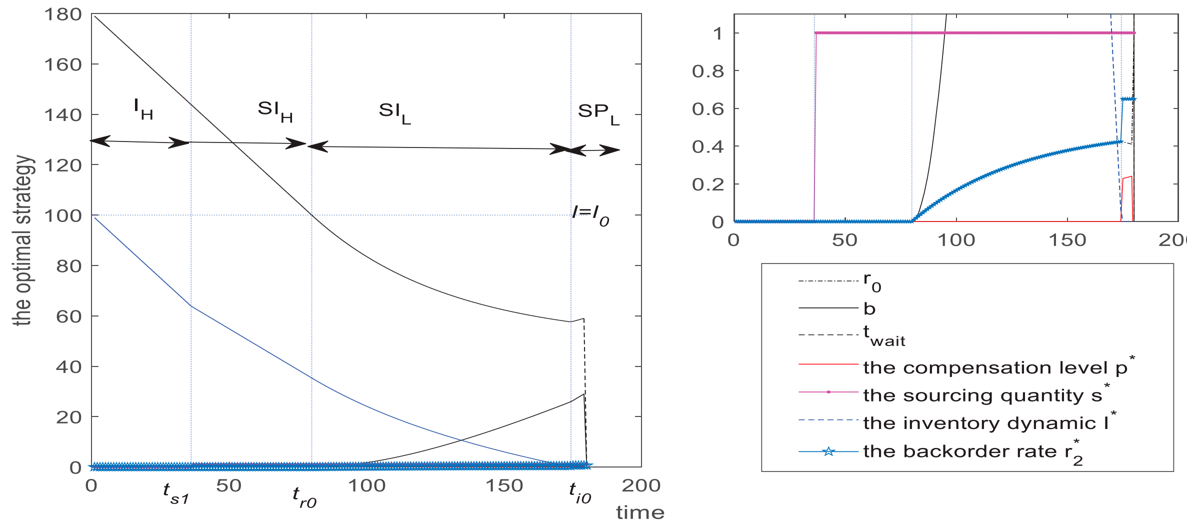
Next, by taking the strategies $I_H - SI_H - SP_H$ and $SI_H - SP_H$as examples, we address a comparison analysis between our proposed dynamic mixed strategies and other pure strategies. To this end, we consider three commonly utilised pure strategies (as summarised earlier in our overview of the existing literature): $I - P$, $I - S$, and $I - N$, indicating the following countermeasures during disruption. First, using inventory to satisfy all demands. Then, compensating customers at the level of '1' and backlogging all demands (i.e. $I - P$), contingent sourcing to satisfy customers and letting the ones who reject this policy be lost sales (i.e. $I - S$), or doing nothing and losing all sales (i.e. $I - N$). Figure 8 depicts the disruption impact ΔC under these strategies and our proposed strategies.

Two findings are observed in Figure 8. First, by using the dynamic mixed strategy $I_H - SI_H - SP_H$, the manufacturer can benefit from stock-outs under some circumstances. The reason is as follows. The benchmark of evaluating the disruption impact is the total cost of the manufacturer carrying I_0 units of safety stocks when no disruption occurs. Therefore, an interesting case might occur. If the compensation cost is small enough, the manufacturer can satisfy customers at a lower cost than holding safety inventory would have cost him. Second, compared with the other three pure strategies, by using optimal mixed strategies during the first 29 days to manage a 32-days disruption, the loss can be reduced by 50% at the minimum and 86% at the maximum(Figures 8(c)-(d)), unless the compensation cost or the contingent sourcing cost is extremely small (Figures 8(a)-(b)). We also observe that the reduced loss through these two mixed strategies increases with the disruption length. The result confirms the importance of developing such combined strategies when hedging against disruptions.

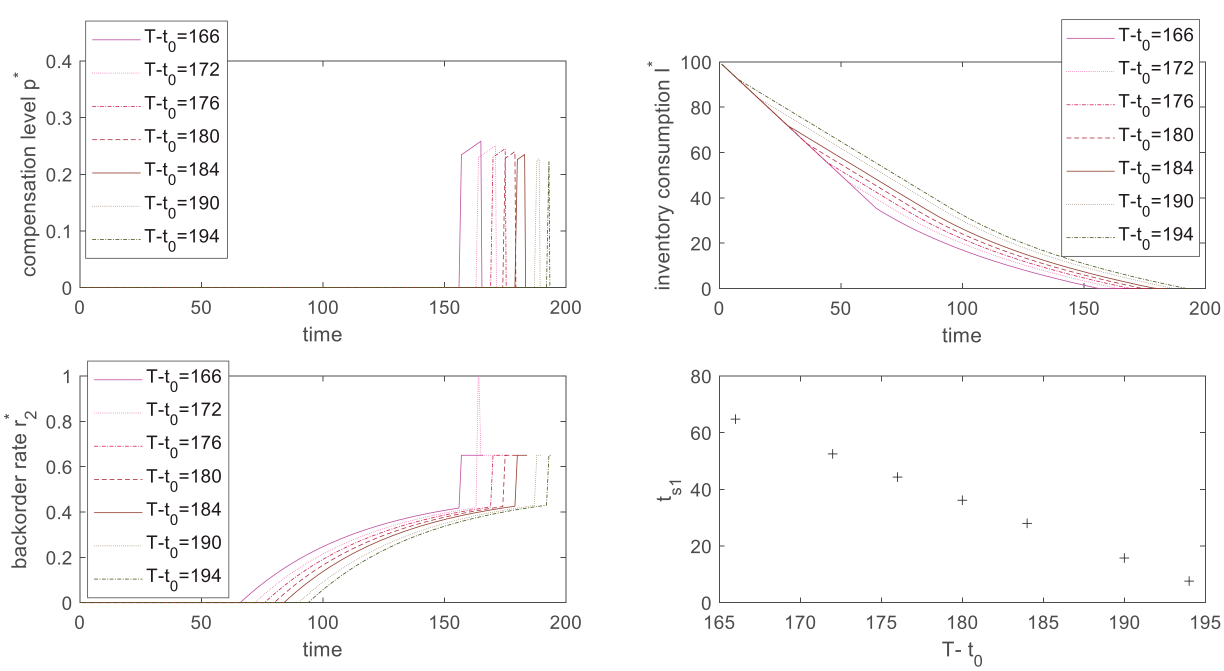
5. Managerial insights

In this paper, we consider a make-to-order system where demand is deterministic but sensitive to both the price and the delivery time. The manufacturer procures from a single unreliable supplier and carries a certain amount of raw materials as a precaution, while there is an emergency supplier with a high price and a lead time in the market. By incorporating proactive inventory consumption, reactive supply-side sourcing, and demand-side compensation, we analytically investigate the optimal joint dynamic disruption-management decision for the manufacturer, taking customers' dynamic state-dependent backordering behaviour into consideration. Our results render the following insights for decision-makers.

(1) When the system holds positive stocks: no compensation should be offered to customers. However, contingent replenishments should be taken in conjunction with inventory consumption, i.e. utilising the combination strategies SI_H and SI_L during those



(a) The decisions of $I_H - SI_H - SI_L - SP_L$ under $T - t_0 = 180$.



(b) The $I_H - SI_H - SI_L - SP_H$ under different disruption length T.

Figure 7. The optimal dynamic strategy $I_L - SI_H - SI_L - SP_H$ under $c_s = 15$, $c_p = 45$ and different T.

periods when the sourcing cost is less than the shadow price of utilising safety inventory. Furthermore, in the process of practicing these two combination strategies, the contingent sourcing strategy should always be first offered to customers. We only consider satisfying those customers who reject to wait for contingent sourcing through inventory.

(2) At the occurrence of disruption, there are four options to choose from for the initial strategy: the pure inventory consumption strategies I_H and I_L, and the combination strategies SI_H and SI_L, mainly determined by the disruption length and the initial value (price) of backordering and consuming inventory. It is not always advisable to consume inventory to meet all demands. If the disruption will not last long and the economic value of holding a unit backorder exceeds the shadow price of a unit inventory, it is preferable for the manufacturer to accept backorders without providing compensation. The contingent sourcing policy should be employed from

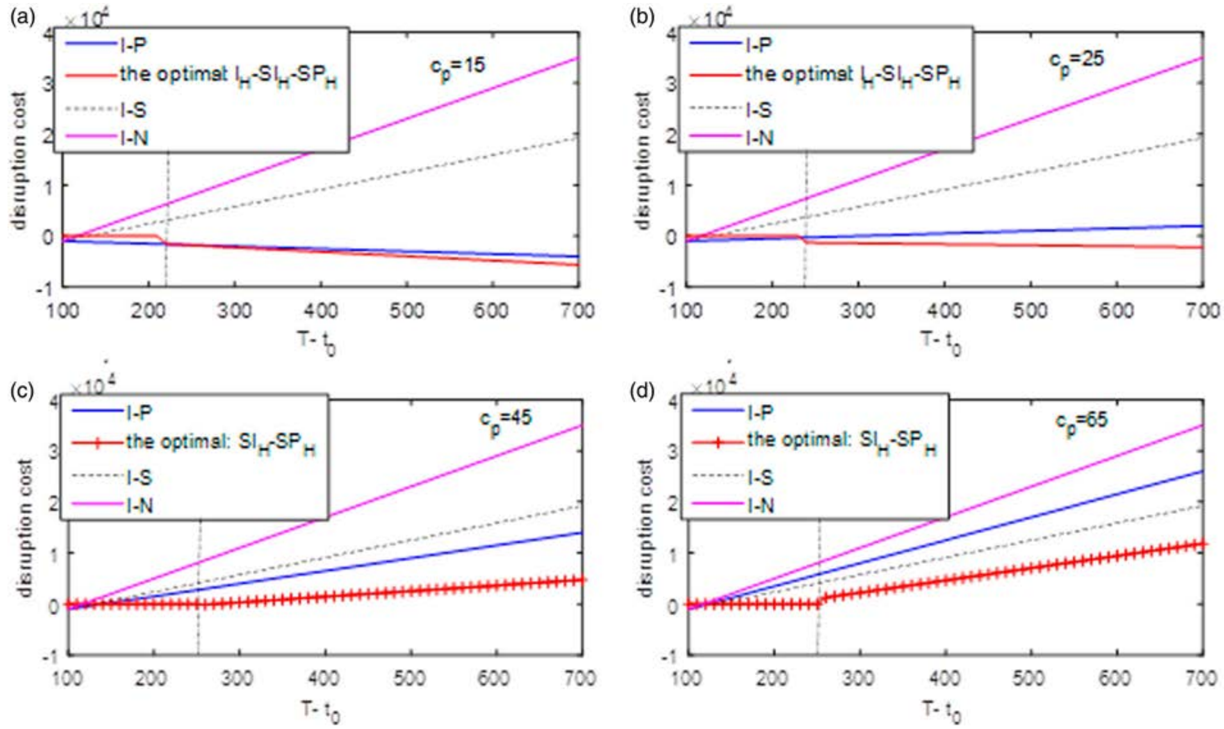


Figure 8. The impact differences between utilising pure strategies and mixed strategies for long disruptions.

the outset if it shows a significant advantage in terms of cost and lead time.

(3) At the end in time when the contingent sourcing is available, there are seven options from which to choose for a termination strategy: the pure sourcing strategies S_H^0, S_H^r, and S_L, the combination strategies SP_H and SP_L, the pure compensation strategies P_H and P_L, mainly determined by the cost factors of sourcing and compensation.

(4) Cost and time factors play different roles when designing optimal dynamic mitigation strategies. The cost factors mainly determine the time intervals for contingent sourcing, demand backlogging, and customer compensation. The question of how to dynamically adjust the backorder rate, sourcing quantity, compensation level, and inventory consumption rate is essentially determined by time factors such as the lead time of sourcing, customer sensitivity to waiting time, and the disruption duration.

(5) Only when compensation (or contingent sourcing) is significantly advantageous in terms of cost (cost and lead time), it is optimal to purely compensate customers (or contingent sourcing) after the inventory has been used up. Otherwise, the optimal is to construct a dynamic strategy based on our proposed 11 types of strategy components and their corresponding time intervals.

(6) For short disruptions, the optimal dynamic strategies are mainly composed of pure strategies. As for long disruptions, it is optimal to implement the combined strategies that simultaneously incorporate two countermeasures (such as SP_H, SP_L, SI_H, SI_L) during some periods.

6. Conclusions

Contingent sourcing and safety inventory are two commonly employed supply-side strategies. Compensation (such as responsive price discounts) is a well-utilised strategy to further reduce the negative impact of disruption from the demand-side. On the other hand, there is a critically growing interest in both the literature and practice to develop mixed or dynamic strategies that outperform the existing pure (static) strategies, taking customers' dynamic behaviours into consideration. Therefore, we investigate the optimal joint dynamic disruption-management decision for the manufacturer, incorporating compensation, contingent sourcing, and inventory consumption.

We consider the disruption impact from two periods: disruption duration and disruption recovery. During the disruption, four types of customers' reaction are taken into account: the placement of a backorder due to compensation, the placement of a backorder without compensation (the customers' state-dependent backordering behaviour, linked to the state of backlogged demand and their patience on waiting time), the customers are satisfied through inventory in real-time or through contingent souring after lead time, or the customers will not purchase. During recovery, the manufacturer satisfies the accumulated demand in the order of arrival by using his maximum production capacity (the recovery capacity) and refills safety inventory. By capturing the dynamic states of backlogged demand and inventory along with the customers' state-dependent backordering behaviour, an optimal control model is proposed for minimising the disruption impact.

By using the Pontryagin's Maximum Principle, 11 types of optimal strategies are proposed in closed form. They help with the following joint decision: the dynamic compensation level, the dynamic quantity of contingent sourcing, and the dynamic speed at which safety inventory is consumed. It is worth noting that the results also indicate that the contingent sourcing policy is always announced to customers before other policies in the process of realising the combination strategies. The conditions under which the manufacturer should implement these strategies are presented as well. The transition conditions indicating how to change the above strategies according to the state of demand and inventory are also presented in closed form. They provide analytical guidance on how to jointly adjust the optimal strategy from the initial time of the supply disruption. Furthermore, via numerical analysis, we compare the established strategies with other alternative strategies and visually indicate how much disruption loss can be reduced under our strategies.

This study suggests several directions for future research. One possibility is the consideration of long-term losses in the design of mitigation strategies, for instance, fewer future orders. Another idea is the incorporation of partial backordering behaviour and interaction among customers. In addition to time and price, the customers' interaction through online reviews and social networking also significantly affects their reactions to disruptions. Third, our model is limited to a pre-disruption deterministic demand and a deterministic disruption. An interesting and valuable research direction would be to analyse the problem in the presence of random demand or random disruptions.

Acknowledgements

This work was done when the first author stayed at the Technical University of Munich as a visiting Ph.D. The authors sincerely thank the editors and three anonymous referees for their numerous insightful comments that greatly improved the paper.

This work is supported by National Nature Science Foundation under Grant (Nos.71771053, 71371003 and 72001113) and the Key Research and Development Plan (Modern Agriculture) of Jiangsu Province (No. BE2018385).

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by National Natural Science Foundation of China: [Grant Number 71371003, 71771053, 72001113]; Key Research and Development Plan (Modern Agriculture) of Jiangsu Province, China: [Grant Number BE2018385]; Applied Economics of Nanjing Audit University of the Priority Academic Program Development of Jiangsu Higher Education Institutions, China: [Grant Number [2018] 87 (Office of Jiangsu Provincial People's Government)].

Notes on contributors

Shanshan Li received the Ph.D. degree in Management Science and Engineering from the School of Economics and Management, Southeast University (China). She is currently a Lecturer at the School of Finance, Nanjing Audit University. Her research interests include supply chain risk management, operations planning and control, and supply chain financing.

Yong He is a professor at School of Economics and Management, Southeast University, China. His research interests include supply chain management, logistics management, marketing/OM interfaces, food supply chain, and service science.

Stefan Minner is a Full Professor for Logistics and Supply Chain Management at the School of Management, Technical University of Munich (TUM). Currently, Stefan Minner is the Editor-in-Chief of the International Journal of Production Economics. His research interests are in business analytics, global supply chain design, transportation optimisation and inventory management.

ORCID

Shanshan Li http://orcid.org/0000-0002-0704-3770
Yong He http://orcid.org/0000-0002-6834-0883
Stefan Minner http://orcid.org/0000-0001-6127-8223

References

Accenture. 2020. "Impact on the Automotive Industry: Navigating the Human and Business Impact of COVID-19." https://www.accenture.com/_acnmedia/PDF-124/Accenture-COVID-19-Impact-on-the-Automotive-Industry.pdf.

Azad, N., and E. Hassini. 2019. "Recovery Strategies from Major Supply Disruptions in Single and Multiple Sourcing Networks." *European Journal of Operational Research* 275: 481–501.

Baryannis, G., S. Validi, S. Dani, and G. Antoniou. 2019. "Supply Chain Risk Management and Artificial Intelligence: State of the art and Future Research Directions." *International Journal of Production Research* 57: 2179–2202.

Bhargava, H. K., D. Sun, and S. H. Xu. 2006. "Stock out Compensation: Joint Inventory and Price Optimization in Electronic Retailing." *INFORMS Journal on Computing* 18 (2): 255–266.

Breen, B. 2011. "Living in Dell Time". https://www.fastcompany.com/51967/living-dell-time.

Chen, J., S. Huang, R. Hassin, and N. Zhang. 2015. "Two Backorder Compensation Mechanisms in Inventory Systems with Impatient Customers." *Production and Operations Management* 24 (10): 1640–1656.

Chen, K. B., and L. Yang. 2014. "Random Yield and Coordination Mechanisms of a Supply Chain with Emergency Backup Sourcing." *International Journal of Production Research* 52: 4747–4767.

Ding, Q., P. Kouvelis, and J. M. Milner. 2006. "Dynamic Pricing Through Discounts for Optimizing Multiple-Class Demand Fulfillment." *Operations Research* 54 (1): 169–183.

Dong, Y., K. Xu, T. H. Cui, and Y. Yao. 2015. "Service Failure Recovery and Prevention: Managing Stockouts in Distribution Channels." *Marketing Science* 34 (5): 689–701.

Drake, M. J., and D. W. Pentico. 2011. "Price Discounts for Increased Profitability Under Partial Backordering." *International Transactions in Operational Research* 18 (1): 87–101.

Fortune. 2020. https://fortune.com/2020/02/21/fortune-1000-coronavirus-china-supply-chain-impact/, accessed on March 10, 2020.

Gao, L. 2015. "Collaborative Forecasting, Inventory Hedging and Contract Coordination in Dynamic Supply Risk Management." *European Journal of Operational Research* 245: 133–145.

Golmohammadi, A., and E. Hassini. 2020. "Review of Supplier Diversification and Pricing Strategies Under Random Supply and Demand." *International Journal of Production Research* 58 (11): 3455–3487.

Gupta, V., B. He, and S. P. Sethi. 2015. "Contingent Sourcing Under Supply Disruption and Competition." *International Journal of Production Research* 53 (10): 3006–3027.

Gupta, V., and D. Ivanov. 2020. "Dual Sourcing Under Supply Disruption with Risk-Averse Suppliers in the Sharing Economy." *International Journal of Production Research* 58: 291–307.

Gupta, V., D. Ivanov, and T. M. Choi. 2020. "Competitive Pricing of Substitute Products Under Supply Disruption." *Omega*. doi:10.1016/j.omega.2020.102279.

He, J., F. Alavifard, D. Ivanov, and H. Jahani. 2019. "A Real-Option Approach to Mitigate Disruption Risk in the Supply Chain." *Omega* 88: 133–149.

He, B., H. Huang, and K. Yuan. 2016. "Managing Supply Disruption Through Procurement Strategy and Price Competition." *International Journal of Production Research* 54 (7): 1980–1999.

He, Y., S. Li, H. Xu, and C. Shi. 2020. "An In-Depth Analysis of Contingent Sourcing Strategy for Handling Supply Disruptions." *IEEE Transactions on Engineering Management* 67 (1): 201–219.

Ivanov, D., and A. Dolgui. 2019. "Low-Certainty-Need (LCN) Supply Chains: a new Perspective in Managing Disruption Risks and Resilience." *International Journal of Production Research* 57: 5119–5136.

Ivanov, D., A. Dolgui, and B. Sokolov. 2018. "Scheduling of Recovery Actions in the Supply Chain with Resilience Analysis Considerations." *International Journal of Production Research* 56 (19): 6473–6490.

Ivanov, D., A. Dolgui, B. Sokolov, and M. Ivanova. 2017. "Literature Review on Disruption Recovery in the Supply Chain." *International Journal of Production Research* 55 (20): 6158–6174.

Ivanov, D., A. A. Pavlov, A. Dolgui, D. Pavlov, and B. Sokolov. 2016. "Disruption-driven Supply Chain (re)-Planning and Performance Impact Assessment with Consideration of pro-Active and Recovery Policies." *Transportation Research Part E-Logistics and Transportation Review* 9: 7–24.

Kouvelis, P., and J. Li. 2012. "Contingency Strategies in Managing Supply Systems with Uncertain Lead-Times." *Production and Operations Management* 21 (1): 161–176.

Kumar, M., P. Basu, and B. Avittathur. 2018. "Pricing and Sourcing Strategies for Competing Retailers in Supply Chains Under Disruption Risk." *European Journal of Operational Research* 265: 533–543.

Li, S., Y. He, and L. Chen. 2017. "Dynamic Strategies for Supply Disruptions in Production-Inventory Systems." *International Journal of Production Economics* 194: 88–101.

Namdar, J., X. P. Li, R. Sawhney, and N. Pradhan. 2018. "Supply Chain Resilience for Single and Multiple Sourcing in the Presence of Disruption Risks." *International Journal of Production Research* 56: 2339–2360.

Paul, S. K., and S. Rahman. 2018. "A Quantitative and Simulation Model for Managing Sudden Supply Delay with Fuzzy Demand and Safety Stock." *International Journal of Production Research* 56 (13): 4377–4395.

Pentico, D. W., C. Toews, and M. J. Drake. 2015. "Approximating the EOQ with Partial Backordering at an Exponential or Rational Rate by a Constant or Linearly Changing Rate." *International Journal of Production Economics* 162: 151–159.

Qi, L. 2013. "A Continuous-Review Inventory Model with Random Disruptions at the Primary Supplier." *European Journal of Operational Research* 225: 59–74.

Qi, L., Z. J. M. Shen, and L. V. Snyder. 2009. "A Continuous-Review Inventory Model with Disruptions at Both Supplier and Retailer." *Production and Operations Management* 18 (5): 516–532.

Rigby, D., and B. Bilodeau. 2009. "Management Tools 2009: An Executive's Guide." https://media.bain.com/Images/Management_Tools_2009.pdf, accessed on November 3, 2020.

Saghafian, S., and M. P. Van Oyen. 2016. "Compensating for Dynamic Supply Disruptions: Backup Flexibility Design." *Operations Research* 64 (2): 390–405.

Saithong, C., and H. T. Luong. 2019. "Effect of Supply Disruption on Inventory Policy." *European Journal of Industrial Engineering* 13 (2): 178–212.

Sarkar, B., B. Mandal, and S. Sarkar. 2015. "Quality Improvement and Backorder Price Discount Under Controllable Lead Time in an Inventory Model." *Journal of Manufacturing Systems* 35: 26–36.

Schmitt, T. G., S. Kumar, K. E. Stecke, F. W. Glover, and M. A. Ehlen. 2017. "Mitigating Disruptions in a Multi-Echelon Supply Chain Using Adaptive Ordering." *Omega* 68: 185–198.

Seierstad, A., and K. Sydsæter. 1987. "Optimal Control Theory with Economic Applications." In *Advanced Textbooks in Economics*, edited by C. J. Bliss and M. D. Intriligator, Vol. 24, 463 *Amsterdam*: North-Holland.

Shao, X. F. 2018. "Production Disruption, Compensation, and Transshipment Policies." *Omega* 74: 37–49.

Shao, X. F., and M. Dong. 2012. "Supply Disruption and Reactive Strategies in an Assemble-to-Order Supply Chain With Time-Sensitive Demand." *IEEE Transactions on Engineering Management* 59 (2): 201–212.

Silbermayr, L., and S. Minner. 2014. "A Multiple Sourcing Inventory Model Under Disruption Risk." *International Journal of Production Economics* 149: 37–46.

Silbermayr, L., and S. Minner. 2016. "Dual Sourcing Under Disruption Risk and Cost Improvement Through Learning." *European Journal of Operational Research* 250 (1): 226–238.

Snyder, L. V., Z. Atan, P. Peng, Y. Rong, A. J. Schmitt, and B. Sinsoysal. 2016. "OR/MS Models for Supply Chain Disruptions: a Review." *IIE Transactions* 48 (2): 89–109.

Su, X., and F. Zhang. 2009. "On the Value of Commitment and Availability Guarantees When Selling to Strategic Consumers." *Management Science* 55 (5): 713–726.

Svoboda, J., S. Minner, and M. Yao. 2020. "Typology and Literature Review on Multiple Supplier Inventory Control Models." SSRN: https://ssrn.com/abstract = 2995134.

Taleizadeh, A. A. 2017. "Lot-Sizing Model with Advance Payment Pricing and Disruption in Supply Under Planned Partial Backordering." *International Transactions in Operational Research* 24 (4): 783–800.

Tomlin, B. 2006. "On the Value of Mitigation and Contingency Strategies for Managing Supply Chain Disruption Risks." *Management Science* 52 (5): 639–657.

Tomlin, B. 2009. "Disruption-Management Strategies for Short Life-Cycle Products." *Naval Research Logistics* 56 (4): 318–347.

Topan, E., and M. C. van der Heijden. 2020. "Operational Level Planning of a Multi-Item two-Echelon Spare Parts Inventory System with Reactive and Proactive Interventions." *European Journal of Operational Research* 284: 164–175.

UKEssays. 2018. "Huawei Supply Chain Operations." https://www.ukessays.com/essays/business/huawei-supply-chain-operations-2826.php?vref = 1.

Wang, Y. J., and Y. G. Yu. 2020. "Flexible Strategies Under Supply Disruption: the Interplay Between Contingent Sourcing and Responsive Pricing." *International Journal of Production Research*. doi:10.1080/00207543.2020.1722326.

Xu, H. 2020. "Minimizing the Ripple Effect Caused by Operational Risks in a Make-to-Order Supply Chain." *International Journal of Physical Distribution & Logistics Management* 50 (4): 381–402. doi:10.1108/IJPDLM-06-2018-0213.

Yadav, N. 2020. "Impact of China's Disrupted Supply Chain on India amid COVID-19." https://www.india-briefing.com/news/covid-19-india-impact-supply-chain-china-19724.html/.

Online Supplement

Appendix

Part A presents the calculation of Table 4, Parts B-C presents the calculation of Table 5, Part D presents the proof of Figures 2-3, and Part E presents the calculation of Table 6 and the proof of Corollary 1.

Part A: Calculation of Table 4

In Part A, we determine the optimal strategies in Inventory Stage 1 with two steps. In step 1, we maximize (33) to achieve $s^*, {r_2}^*, p^*$ under $d^* = 1 - r_2 > r_1 s$. The optimal strategies without losing sales (that is, under the constraint of $d^* = 1 - r_2$) are established. The optimal paths corresponding to these decisions, describing the dynamics of backlogged demand and inventory, are also given in closed forms. In step 2, We analyze the necessary conditions (20)-(31) that each decision $\{s^*, {r_2}^*, p^*, d^* = 1 - r_2\}$ requires to be optimal. The time intervals of each optimal strategy, defined by the necessary conditions in Table A5, are finally achieved for Table 4.

(i) The optimal $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = 1 - r_2 > r_1 s$

According to (32), together with $d > r_1 s$, the optimal s^* is achieved as $s^* = 1$ or $s^* = 0$. Based on (33), $\partial H/\partial s = (\lambda_2 - c_s) r_1$. Thus,

$$s^* = 1 \text{ if } \lambda_2 > c_s, \text{ and } s^* = 0 \text{ if } \lambda_2 < c_s. \quad \text{(A1)}$$

On the other hand, $\partial H/\partial r_2 = \lambda_1 + \lambda_2 - c_0$ indicates that

$${r_2}^* = r_0 + p \ \text{ if } \ \lambda_1 + \lambda_2 > c_0, \text{ and } \ {r_2}^* = 0 \ \text{ if } \ \lambda_1 + \lambda_2 < c_0. \quad \text{(A2)}$$

Substituting ${r_2}^* = r_0 + p$ into (30), the Hamiltonian function is derived as

$$H = (\lambda_1 + \lambda_2 - c_0) r_0 + (\lambda_1 + \lambda_2 - c_0 - c_p) p + (\lambda_2 - c_s) r_1 s - \lambda_2 - c_h I + c_0 + c_h I_0. \quad \text{(A3)}$$

Accordingly, we have $\partial H/\partial p = \lambda_1 + \lambda_2 - c_0 - c_p$. Hence, the optimal control p^* to maximize (A3) is

$$p^* = 1 - r_0 \text{ if } \ \lambda_1 + \lambda_2 > c_p + c_0, \ \text{ and } \ p^* = 0 \text{ if } \lambda_1 + \lambda_2 < c_p + c_0. \quad \text{(A4)}$$

Summing up, and substituting ${r_2}^*$ into d^*, the Hamiltonian function (33) is maximized at the following $\{s^*, {r_2}^*, p^*, d^*\}$, as shown in Table A1. The inventory consumption rate $-\dot{I}^*$ is accordingly calculated.

Table A1. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ and the corresponding $-\dot{I}^*$ under $d^* = 1 - r_2 > r_1 s$.

Scenario	s^*	${r_2}^*$	p^*	d^*	$-\dot{I}^*$	The condition	
A(i)	0	0	0	1	1	$\lambda_2 < c_s$	$\lambda_1 + \lambda_2 < c_0$
A(ii)		r_0	0	$1 - r_0$	$1 - r_0$		$c_0 < \lambda_1 + \lambda_2 < c_0 + c_p$
A(iii)		1	$1 - r_0$	0	0		$\lambda_1 + \lambda_2 > c_0 + c_p$
A(iv)	1	0	0	1	$1 - r_1$	$\lambda_2 > c_s$	$\lambda_1 + \lambda_2 < c_0$
A(v)		r_0	0	$1 - r_0$	$1 - r_0 - r_1$		$c_0 < \lambda_1 + \lambda_2 < c_0 + c_p$
A(vi)		1	$1 - r_0$	0	< 0		$\lambda_1 + \lambda_2 > c_0 + c_p$

In view of $d^* = 1 - r_2 > r_1 s$, we see that the inventory decreases over time during Inventory Stage 1. Therefore, Scenarios A(iii) and A(vi) cannot happen. Scenario A(v) requires an extra condition

$$1 - r_0 > r_1. \tag{A5}$$

Considering that the backorder rate without compensation could be $r_0 = 0$ or $r_0 > 0$, the optimal controls are further specified as in Table A2. Four types of strategies are developed.

Table A2. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ and the corresponding strategy in Inventory Stage 1.

r_0	Scenario	s^*	${r_2}^*$	p^*	d^*	$-\dot{I}^*$	The notation of the strategy
0	A(i)(ii)	0	0	0	1	1	I_H
	A(iv)(v)	1	0	0	1	$1 - r_1$	SI_H
+	A(i)	0	0	0	1	1	I_H
	A(ii)	0	r_0	0	$1 - r_0$	$1 - r_0$	I_L
	A(iv)	1	0	0	1	$1 - r_1$	SI_H
	A(v)	1	r_0	0	$1 - r_0$	$1 - r_0 - r_1$	SI_L

Where, strategy SI_L only happens if the customers' reaction falls into the cases with $0 < r_0 < 1 - r_1$.

Next, we present how the backlogged demands accumulate during the implementation of the above strategies. As indicated by state equation (6), backorders are accumulated at the rate ${r_2}^*$ during the use of mitigation strategies. Therefore, the optimal path b^* capturing the total backlogged demand is described by two expressions in Inventory Stage 1.

(a) The b^* under ${r_2}^* = 0$

That is, $db^*/dt = 0$. No backorder arises, thus, the optimal path b^* remains constant.

$$b^* = b^*(\tau_0), \text{ denoted as } f_1. \quad \text{(A6)}$$

τ_0 is the entry point of the time interval with $r_2{}^* = 0$.

(b) The $\boldsymbol{b^*}$ under $\boldsymbol{r_2{}^* = r_0 > 0}$

Given $r_2{}^* = r_0 = 1 - \theta(T + \frac{b^*}{P_{\mathrm{C}-1}} - t) > 0$, together with the state equation (6), b^* is identified by the linear differential equation

$$db^*/dt = 1 - \theta(T + \frac{b^*}{P_{\mathrm{C}-1}} - t). \quad \text{(A7)}$$

Solving this equation, b^* is achieved as

$$b^* = C_0\, e^{-\frac{\theta t}{P_{\mathrm{C}-1}}} + \frac{P_{\mathrm{C}}-1}{\theta}(2 - P_{\mathrm{C}} - \theta T + \theta t), \text{ denoted as } f_2, \quad \text{(A8)}$$

where C_0 is a constant to be determined by the initial value of b at the entry point τ_1 to the time interval of $r_2{}^* = r_0$. According to (A8), we see that

$$b^*(\tau_1) = C_0\, e^{-\frac{\theta \tau_1}{P_{\mathrm{C}-1}}} + \frac{P_{\mathrm{C}}-1}{\theta}(2 - P_{\mathrm{C}} - \theta T + \theta \tau_1).$$

Therefore,

$$C_0 = e^{\frac{\theta \tau_1}{P_{\mathrm{C}-1}}}[b^*(\tau_1) - \frac{P_{\mathrm{C}}-1}{\theta}(2 - P_{\mathrm{C}} - \theta T + \theta \tau_1)]. \quad \text{(A9)}$$

(ii) The conditions for $\{\boldsymbol{s^*, r_2{}^*, p^*, d^*}\}$ under $\boldsymbol{d^* = 1 - r_2 > r_1 s}$

For each optimal solution $\{s^*, r_2{}^*, p^*, d^* = 1 - r_2\}$, the necessary conditions (20)-(24) regarding the co-state variables λ_1 and λ_2 must be met. By exploring these necessary conditions, together with the conditions given in Table A1, the time intervals for each optimal strategy defined by λ_1 and λ_2 can be deduced. However, as indicated in (20)-(24), the Lagrangian multipliers $\mu_1,\ldots,\mu_8$ are incorporated into the dynamics of these two co-state variables. Therefore, we next discuss both the necessary conditions and the complementary slackness conditions to determine the time intervals for each optimal strategy.

(a) The necessary conditions for $\boldsymbol{\lambda_1}$

The equations linked to λ_1 are (20), (21), and (24). First, according to (20), we find: $\lambda_1 > c_0 + c_1 - c_l$ if $-\mu_2 + \mu_5 - \mu_6 < 0$; and $\lambda_1 < c_0 + c_1 - c_l$ if $-\mu_2 + \mu_5 - \mu_6 >$

0. Then, combining (19) and (21), the derivative $\frac{d\lambda_1}{dt}$ of the co-state variable λ_1 under $r_0 > 0$ can be derived as $\dot{\lambda_1} = \mu_7 - c_p = -\mu_6 + \mu_8$.

(b) The necessary conditions for λ_2

(22) and (23) give the necessary conditions for λ_2: $\lambda_2 < c_l - c_1$ if $\mu_1 - \mu_2 < 0$; and $\lambda_2 > c_l - c_1$ if $\mu_1 - \mu_2 > 0$. On the other hand, $\lambda_2 > c_s$ if $-\mu_1 + \mu_3 - \mu_4 < 0$; and $\lambda_2 < c_s$ if $-\mu_1 + \mu_3 - \mu_4 > 0$.

Summing up, the necessary conditions derived from (20)-(24) are stated in Table A3.

Table A3. The necessary conditions derived from (20)-(24).

The conditions of λ_1 and λ_2	The corresponding multipliers
$\lambda_1 > c_0 + c_1 - c_l$	$-\mu_2 + \mu_5 - \mu_6 < 0$
$\lambda_2 < c_l - c_1$	$\mu_1 - \mu_2 < 0$
$\lambda_2 > c_s$	$-\mu_1 + \mu_3 - \mu_4 < 0$
$\dot{\lambda_1} = 0$	if $r_0 = 0$
$\dot{\lambda_1} = \mu_7 - c_p = -\mu_6 + \mu_8$	if $r_0 > 0$

As shown in Table A3, the following two questions need to be addressed in order to derive the time intervals for each optimal decision: First, we need to specify λ_1 from the derivative $\dot{\lambda_1}$, which is related to $\mu_7 - c_p$ and $-\mu_6 + \mu_8$ if $r_0 > 0$. Then, we have to generate the time intervals from the conditions of the co-state variables, which are linked to the positivity of the following items: $-\mu_2 + \mu_5 - \mu_6$, $\mu_1 - \mu_2$, and $-\mu_1 + \mu_3 - \mu_4$. Therefore, we next identify the multipliers μ_1,…, μ_8 for each $\{s^*, {r_2}^*, p^*, d^*\}$.

(c) The conditions for μ_1,…, μ_8

Based on the complementary slackness conditions (28)-(31), the positivity of the multipliers μ_1,…, μ_8 can be initially identified. Accordingly, we can directly deduce the items $\mu_7 - c_p$ and $-\mu_6 + \mu_8$ for the cases with $r_0 > 0$, and the positivity of the aforementioned items $-\mu_2 + \mu_5 - \mu_6$, $\mu_1 - \mu_2$, and $-\mu_1 + \mu_3 - \mu_4$. The results are given in Table A4.

As stated in Table A4, the co-state variable λ_1 has been set to be a constant in Scenarios A(i) and A(iv) with $r_0 > 0$. Nonetheless, the values of $\mu_7 - c_p$ and $-\mu_6 + \mu_8$ are still undetermined in Scenarios A(ii) and A(v) with $r_0 > 0$. As a result, the co-state variable λ_1 is still undetermined. The following property can be observed in these two scenarios.

$$-\dot{\lambda}_1 = \frac{\theta}{P_C - 1}\mu_6 = \frac{\theta}{P_C - 1}\mu_6 = \frac{\theta}{P_C - 1}\left(c_p - \mu_7\right) < \frac{\theta}{P_C - 1}c_p. \tag{A10}$$

that is, the co-state variable λ_1 drops at a speed lower than $\frac{\theta}{P_C - 1}c_p$.

Table A4. The corresponding multipliers for $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = 1 - r_2 > r_1 s$.

r_0	Scenario	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	$\mu_7 - c_p$	$-\mu_6 + \mu_8$	$-\mu_2 + \mu_5 - \mu_6$	$\mu_1 - \mu_2$	$-\mu_1 + \mu_3 - \mu_4$
0	A(i)(ii)	0	+	+	0	+	+	+	0	No	No	U	−	+
0	A(iv)(v)	0	+	0	+	+	+	+	0	No	No	U	−	−
+	A(i)	0	+	+	0	+	0	+	0	$\mu_7 - c_p$	0	U	−	+
+	A(ii)	0	+	+	0	0	+	+	0	$\mu_7 - c_p$	$-\mu_6$	−	−	+
+	A(iv)	0	+	0	+	+	0	+	0	$\mu_7 - c_p$	0	U	−	−
+	A(v)	0	+	0	+	0	+	+	0	$\mu_7 - c_p$	$-\mu_6$	−	−	−

"U" means that the positivity is undetermined; "No" means that there is no need to consider this item.

Next, by further discussing (20)-(23) and determining the value of μ_6, we identify λ_1 in Scenarios A(ii) and A(v) with $r_0 > 0$.

Given $\mu_1 = \mu_5 = 0$ in Scenarios A(ii) and A(v), (20) and (22) are generated as

$$\partial L/\partial r_2 = c_l - (c_0 + c_1) + \lambda_1 - \mu_2 - \mu_6 = 0, \text{ and}$$

$$\partial L/\partial d = -c_1 + c_l - \lambda_2 - \mu_2 = 0.$$

μ_6 is then determined as $\mu_6 = \lambda_1 + \lambda_2 - c_0$. Therefore, (24) is derived as

$$\dot{\lambda}_1 = \frac{-\theta}{P_C - 1}(\lambda_1 + \lambda_2 - c_0), \tag{A11}$$

where $\lambda_2 = c_h t + \lambda_2(0)$. Solving this linear differential equation, λ_1 is achieved as

$$\lambda_1 = C_1 e^{-\frac{\theta}{(P_C - 1)}t} + c_0 - \lambda_2(0) + c_h \frac{P_C - 1}{\theta} - c_h t, \text{ denoted as } F_1. \tag{A12}$$

C_1 is a constant determined by the initial value $\lambda_1(\tau_{F1})$ at the entry time τ_{F1} of either strategy I_L or strategy SI_L.

$$\lambda_1(\tau_{F1}) = C_1 e^{-\frac{\theta \tau_{F1}}{(P_C - 1)}} + c_0 - \lambda_2(0) + c_h \frac{P_C - 1}{\theta} - c_h \tau_{F1}. \leftrightarrow$$

$$C_1 = [\lambda_1(\tau_{F1}) - c_0 + \lambda_2(0) - c_h \frac{P_C - 1}{\theta} + c_h \tau_{F1}] e^{\frac{\theta \tau_{F1}}{(P_C - 1)}}. \tag{A13}$$

After specifying the co-state variable λ_1, given Table A4 together with Table A3, the conditions of the above optimal decisions, which define the corresponding time intervals, are shown in Table A5.

Table A5. The conditions of optimal decisions in Inventory Stage 1.

r_0	Scenario	The co-state variable λ_1		The conditions of optimal decisions	
		$-\dot{\lambda}_1$	λ_1	$\lambda_2 < c_l - c_1$	$\lambda_1 + \lambda_2 < c_0 + c_p$
0	A(i)(ii)	0	constant	$\lambda_2 < c_s$	
0	A(iv)(v)	0	constant	$\lambda_2 > c_s$	
+	A(i)	0	constant	$\lambda_2 < c_s$	$\lambda_1 + \lambda_2 < c_0$
+	A(ii)	$< \frac{\theta}{P_C - 1} c_p$	F_1	$\lambda_2 < c_s$	$\lambda_1 + \lambda_2 > c_0$
+	A(iv)	0	constant	$\lambda_2 > c_s$	$\lambda_1 + \lambda_2 < c_0$
+	A(v)	$< \frac{\theta}{P_C - 1} c_p$	F_1	$\lambda_2 > c_s$	$\lambda_1 + \lambda_2 > c_0$

where F_1 is given in (A12).

Synthesizing the above results in Tables A1 and A5, the optimal decisions and the corresponding conditions in Inventory Stage 1 are generated in Table 4. □

Calculation of Table 5

In the following chapter, Part B and Part C present the calculation that determines the optimal decisions under $d^* = r_1 s = 1 - r_2$ and $d^* = r_1 s < 1 - r_2$.

Part B: The optimal decisions derived from $\boldsymbol{d^* = r_1 s < 1 - r_2}$

Similar to the analysis in Part A, the optimal decisions are achieved in two steps: First, we determine the optimal values of the joint control $\{s^*, {r_2}^*, p^*, d^*\}$. Then, we present the conditions for all developed optimal decisions, that is, the time intervals for each optimal strategy.

(i) The optimal values of $\boldsymbol{\{s^*, {r_2}^*, p^*, d^*\}}$ under $\boldsymbol{d^* = r_1 s < 1 - r_2}$

According to (35), s^* and ${r_2}^*$ are determined as

$$s^* = 1 \text{ if } c_1 + c_s < c_l, \text{ and } s^* = 0 \text{ if } c_1 + c_s > c_l. \quad \text{(B1)}$$

$${r_2}^* = r_0 + p \;\text{ if }\; \lambda_1 > c_0 + c_1 - c_l, \text{ and } \;{r_2}^* = 0 \;\text{ if }\; \lambda_1 < c_0 + c_1 - c_l. \quad \text{(B2)}$$

Substituting ${r_2}^* = r_0 + p$ into (35), the Hamiltonian function is derived as

$$H = [c_l - (c_0 + c_1) + \lambda_1](r_0 + p) - c_p p + (-c_1 - c_s + c_l - \lambda_2) r_1 s - c_l + (c_0 + c_1 + c_h I_0). \quad \text{(B3)}$$

p^* to maximize (B3) is determined as

$$p^* = 1 - r_0 \text{ if } \lambda_1 > c_0 + c_1 - c_l + c_p, \text{ and } p^* = 0 \text{ if } \lambda_1 < c_0 + c_1 - c_l + c_p. \quad \text{(B4)}$$

Summing up the above analysis and substituting s^* into d^*, we see that under $d^* = r_1 s < 1 - r_2$, $\{s^*, {r_2}^*, p^*, d^*\}$ is obtained in Table B1.

Table B1. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = r_1 s < 1 - r_2$.

Scenario	s^*	${r_2}^*$	p^*	d^*	The condition	
B(i)	0	0	0	0	$c_1 + c_s > c_l$	$\lambda_1 < (c_0 + c_1) - c_l$
B(ii)		r_0	0			$(c_0 + c_1) - c_l < \lambda_1 < (c_0 + c_1) - c_l + c_p$
B(iii)		1	$1 - r_0$			$\lambda_1 > (c_0 + c_1) - c_l + c_p$
B(iv)	1	0	0	r_1	$c_1 + c_s < c_l$	$\lambda_1 < (c_0 + c_1) - c_l$
B(v)		r_0	0			$(c_0 + c_1) - c_l < \lambda_1 < (c_0 + c_1) - c_l + c_p$
B(vi)		1	$1 - r_0$			$\lambda_1 > (c_0 + c_1) - c_l + c_p$

In this study, we focus on the disruption events with $c_1 + c_s < c_l$, thus Scenarios B(i)- B(iii) can be excluded. Furthermore, due to $r_1 s < 1 - r_2$, Scenario B(vi) cannot happen, and Scenario B(v) requires an extra condition $1 - r_0 > r_1$. Therefore, taking the sub-scenarios of the customers' reaction into consideration, Table B2 is derived.

Table B2. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = r_1 s < 1 - r_2$ and different r_0.

r_0	Scenario	s^*	${r_2}^*$	p^*	d^*	The notation of the strategy
0	B(iv)(v)	1	0	0	r_1	S_H^0
$0 < r_0 < 1 - r_1$	B(iv)	1	0	0	r_1	S_H^0
	B(v)	1	r_0	0	r_1	S_H^r
$r_0 > 1 - r_1$	B(iv)	1	0	0	r_1	S_H^0

(ii) The conditions of the optimal controls under $d^* = r_1 s < 1 - r_2$

According to Table A3, we have to identify the multipliers from the complimentary slackness conditions (28)-(31) in order to derive the time intervals for each optimal strategy. Similar to Table A4, the results for the optimal decisions under $d^* = r_1 s < 1 - r_2$ are shown in Table B3.

Table B3. The multipliers for $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = r_1 s < 1 - r_2$.

r_0	Scenario	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	$\mu_7 - c_p$	$-\mu_6 + \mu_8$	$-\mu_2 + \mu_5 - \mu_6$	$\mu_1 - \mu_2$	$-\mu_1 + \mu_3 - \mu_4$
0	B(iv)(v)	$+$	0	0	$+$	$+$	$+$	$+$	0	No	No	U	$+$	$-$
$+$	B(iv)			0	$+$	$+$	0			$\mu_7 - c_p$	0	$+$	$+$	$-$
	B(v)			0	$+$	0	$+$			$\mu_7 - c_p$	$-\mu_6$	$-$	$+$	$-$

As shown in Table B3, the co-state variable λ_1 needs to be further discussed for Scenario B(v). Given $\mu_2 = \mu_5 = 0$ in Scenario B(v), μ_6 is identified from (20) as $\mu_6 = c_l - (c_0 + c_1) + \lambda_1$. Therefore, (24) defining λ_1 is derived as

$$\frac{d\lambda_1}{dt} + \frac{\theta}{P_C - 1}[c_l - (c_0 + c_1) + \lambda_1] = 0. \tag{B5}$$

Similar to (A11), by solving the above linear differential equation, λ_1 is determined in the above Scenario B(v). That is, the λ_1 corresponding to the strategy S_H^r is determined as

$$\lambda_1 = C_2 \exp\{-\frac{\theta t}{(P_C - 1)}\} + c_0 + c_1 - c_l, \text{ denoted as } F_2;$$

$$C_2 = [\lambda_1(\tau_{F2}) - c_0 - c_1 + c_l] \exp\{\frac{\theta \tau_{F2}}{(P_C - 1)}\}. \tag{B6}$$

τ_{F2} is the entry point of the time interval under the use of strategy S_H^r.

Given λ_1, and combining the inequalities presented in Table A3 with the results in Table B3, together with Table B2, the conditions under which the optimal controls under $d^* = r_1 s < 1 - r_2$ are the optimal decision in Inventory Stage 2 are shown in Table B4.

TableB4. The conditions for optimal decisions in Inventory Stage 2 under $d^* = r_1 s < 1 - r_2$.

r_0	Scenario	$\lambda_2 > c_l - c_1$	$-\dot{\lambda}_1$	$\lambda_1 < M_1 + c_p$	λ_1
0	B(iv)(v)	$\lambda_2 > c_s$	0		constant
$+$	B(iv)		0	$\lambda_1 < M_1$	constant
	B(v)		$< \frac{\theta}{P_C - 1} c_p$	$\lambda_1 > M_1$	F_2

Part C: The optimal decisions derived from $\boldsymbol{d^* = r_1 s = 1 - r_2}$

Similarly, the optimal decisions are determined with two steps: the optimal values and the corresponding conditions of $\{s^*, {r_2}^*, p^*, d^*\}$.

(i) The optimal values of $\boldsymbol{\{s^*, {r_2}^*, p^*, d^*\}}$ under $\boldsymbol{d^* = r_1 s = 1 - r_2}$

In view of (8)-(10), the decision variables s and r_2 are limited to

$$0 \le r_1 s \le r_1,\ 1 - r_0 - p \le 1 - r_2 \le 1, \text{ where } 0 \le p \le 1 - r_0.$$

Nonetheless, under $1 - r_2 = r_1 s$, the ranges of these two variables are further identified as (see Figure C1)

$$1 - r_1 \le r_2 \le r_0 + p, \text{ and } (1 - r_0 - p)/r_1 \le s \le 1. \tag{C1}$$

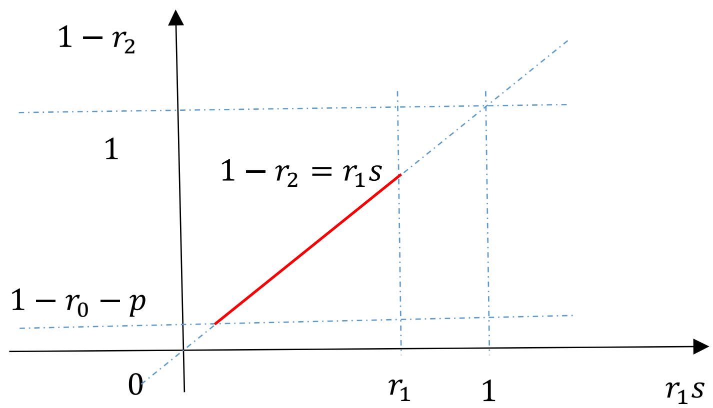


Figure C1. The constraint $1 - r_2 = r_1 s$.

Note that,

$$s = (1 - r_0 - p)/r_1 \le 1 \text{ when } r_2 = r_0 + p, \text{ and } s = 1 \text{ when } r_2 = 1 - r_1. \tag{C2}$$

(C1) also indicates

$$p \ge 1 - r_1 - r_0. \tag{C3}$$

According to (36), $\partial H/\partial r_2 = \lambda_1 + c_s - c_0$. Thus, together with (C1), the optimal $r_2{}^*$ is achieved as

$$r_2{}^* = r_0 + p \ \text{ if } \ \lambda_1 > c_0 - c_s, \text{ and } \ r_2{}^* = 1 - r_1 \ \text{ if } \ \lambda_1 < c_0 - c_s. \tag{C4}$$

Next, we calculate the corresponding compensation level for achieving $r_2{}^*$.

(a) If $r_2{}^* = r_0 + p$:

Substitute $r_2{}^* = r_0 + p$ into (36),

$$H = (\lambda_1 + c_s - c_0)(r_0 + p\ ) - c_p p - c_p p + -c_s - c_h I + c_0 + c_h I_0. \tag{C5}$$

Combining (C3) and $0 \le p \le 1 - r_0$, (C5) is maximized at

$$p^* = 1 - r_0 \text{ if } \lambda_1 > c_0 - c_s + c_p,$$

$$\text{and } \ p^* = \max\{0,\ 1 - r_1 - r_0\} \ \text{ if } \ \lambda_1 < c_0 - c_s + c_p. \tag{C6}$$

(b) If $r_2{}^* = 1 - r_1$:

In view of the customers' reaction, three scenarios are involved if we wish to achieve ${r_2}^* = 1 - r_1 > 0$.

(b-1) When $r_0 > 1 - r_1 > 0$: r_0 customers are willing to backorder without compensation, thus, no compensation is needed, i.e., $p^* = 0$.

(b-2) When $0 < r_0 < 1 - r_1$: The compensation level is determined as $p^* = {r_2}^* - r_0 = 1 - r_1 - r_0 > 0$.

(b-3) When $r_0 = 0$: The compensation level is determined as $p^* = {r_2}^* - r_0 = 1 - r_1 - r_0 = 1 - r_1$.

Table C1. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = 1 - r_2 = r_1 s$.

<table>
<tr><th>s^*</th><th>${r_2}^*$</th><th>p^*</th><th>d^*</th><th colspan="2">The condition</th></tr>
<tr><td rowspan="3">1</td><td rowspan="3">$1 - r_1$</td><td>0</td><td rowspan="5">$1 - r_2$</td><td rowspan="3">$\lambda_1 < c_0 - c_s$</td><td>$r_0 > 1 - r_1$</td></tr>
<tr><td>$1 - r_1 - r_0$</td><td>$0 < r_0 < 1 - r_1$</td></tr>
<tr><td>$1 - r_1$</td><td>$r_0 = 0$</td></tr>
<tr><td rowspan="2">$(1 - r_0 - p)/r_1$</td><td rowspan="2">$r_0 + p$</td><td>$\max\{0,\ 1 - r_1 - r_0\}$</td><td colspan="2">$c_0 - c_s < \lambda_1 < c_0 - c_s + c_p$</td></tr>
<tr><td>$1 - r_0$</td><td colspan="2">$\lambda_1 > c_0 - c_s + c_p$</td></tr>
</table>

Note that r_0 can fall into three sub-scenarios: $r_0 = 0$, $0 < r_0 < 1 - r_1$, and $r_0 > 1 - r_1$. Accordingly, the values of the above optimal controls fall into multiple scenarios. By substituting ${r_2}^*$ into d^*, and p^* into s^*, we present all the scenarios of the optimal controls in Table C2.

Table C2. The optimal controls $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = 1 - r_2 = r_1 s$ and different r_0.

<table>
<tr><th>r_0</th><th>Scenario</th><th>s^*</th><th>${r_2}^*$</th><th>p^*</th><th>d^*</th><th>The notation of the strategy</th></tr>
<tr><td rowspan="2">0</td><td>C(i)</td><td>1</td><td>$1 - r_1$</td><td>$1 - r_1$</td><td>r_1</td><td>SP_H</td></tr>
<tr><td>C(ii)</td><td>0</td><td>1</td><td>1</td><td>0</td><td>P_H</td></tr>
<tr><td rowspan="2">$0 < r_0 < 1 - r_1$</td><td>C(iii)</td><td>1</td><td>$1 - r_1$</td><td>$1 - r_1 - r_0$</td><td>r_1</td><td>SP_L</td></tr>
<tr><td>C(iv)</td><td>0</td><td>1</td><td>$1 - r_0$</td><td>0</td><td>P_L</td></tr>
<tr><td rowspan="3">$r_0 > 1 - r_1$</td><td>C(v)</td><td>1</td><td>$1 - r_1$</td><td>0</td><td>r_1</td><td>S_L^0</td></tr>
<tr><td>C(vi)</td><td>$(1 - r_0)/r_1$</td><td>r_0</td><td>0</td><td>$1 - r_0$</td><td>S_L</td></tr>
<tr><td>C(iv)</td><td>0</td><td>1</td><td>$1 - r_0$</td><td>0</td><td>P_L</td></tr>
</table>

(ii) The conditions of the optimal controls under $d^* = r_1 s = 1 - r_2$

As analyzed in Part A-B, we identify the multipliers $\mu_1, \ldots, \mu_8$ and the corresponding items from (28)-(31), as shown in Table C3.

Table C3. The multipliers for $\{s^*, {r_2}^*, p^*, d^*\}$ under $d^* = 1 - r_2 = r_1 s$.

<table>
<tr><th>Scenario</th><th>μ_1</th><th>μ_2</th><th>μ_3</th><th>μ_4</th><th>μ_5</th><th>μ_6</th><th>μ_7</th><th>μ_8</th><th>μ_7
$-c_p$</th><th>$-\mu_6$
$+\mu_8$</th><th>$-\mu_2$
$+\mu_5$
$-\mu_6$</th><th>μ_1
$-\mu_2$</th><th>$-\mu_1$
$+\mu_3$
$-\mu_4$</th></tr>
<tr><td>C(i)</td><td rowspan="6">$+$</td><td rowspan="6">$+$</td><td>0</td><td>$+$</td><td rowspan="6">0</td><td>$+$</td><td>0</td><td>0</td><td>No</td><td>No</td><td>$-$</td><td>U</td><td>$-$</td></tr>
<tr><td>C(ii)</td><td>$+$</td><td>0</td><td>$+$</td><td>0</td><td>$+$</td><td>No</td><td>No</td><td>$-$</td><td>U</td><td>U</td></tr>
<tr><td>C(iii)</td><td>0</td><td>$+$</td><td>$+$</td><td>0</td><td>0</td><td>$-c_p$</td><td>$-\mu_6$</td><td>$-$</td><td>U</td><td>$-$</td></tr>
<tr><td>C(iv)</td><td>$+$</td><td>0</td><td>$+$</td><td>0</td><td>$+$</td><td>$-c_p$</td><td>$-\mu_6$
$+\mu_8$</td><td>$-$</td><td>U</td><td>U</td></tr>
<tr><td>C(v)</td><td>0</td><td>$+$</td><td>0</td><td>$+$</td><td>0</td><td>μ_7
$-c_p$</td><td>0</td><td>$-$</td><td>U</td><td>$-$</td></tr>
<tr><td>C(vi)</td><td>0</td><td>0</td><td>$+$</td><td>$+$</td><td>0</td><td>μ_7
$-c_p$</td><td>$-\mu_6$</td><td>$-$</td><td>U</td><td>$-$</td></tr>
</table>

According to the values of $\mu_7 - c_p$ and $-\mu_6 + \mu_8$, we see that the co-state variable λ_1 is undetermined in Scenario C(vi). We next identify λ_1 by further discussing (20)-(24). As stated in Table C3, $\mu_3 = \mu_4 = \mu_5 = 0$. Substituting this into (20), (22) and (23), we have

$$\partial L/\partial r_2 = c_l - (c_0 + c_1) + \lambda_1 - \mu_2 - \mu_6 = 0.$$

$$\partial L/\partial d = -c_1 + c_l - \lambda_2 + \mu_1 - \mu_2 = 0.$$

$$\partial L/\partial s = -c_s r_1 + \lambda_2 r_1 - \mu_1 = 0.$$

Solving the above three equations, we have $\mu_6 = \lambda_1 - c_0 + c_s r_1 + \lambda_2(1 - r_1)$. Therefore, together with (24), λ_1 is defined by the equation

$$\frac{d\lambda_1}{dt} + \frac{\theta}{P_C - 1}[\lambda_1 - c_0 + c_s r_1 + \lambda_2(1 - r_1)] = 0. \tag{C7}$$

Similar to (A11), substituting $\lambda_2 = c_h t + \lambda_2(0)$ into the above equation, we have

$\lambda_1 = C_3 \exp\{-\frac{\theta t}{P_C - 1}\} + c_0 - \lambda_2(0)(1 - r_1) - c_s r_1 + c_h(1 - r_1)(\frac{P_C - 1}{\theta} - t)$, denoted as F_3;

$C_3 = \left\{\lambda_1(\tau_{F3}) - \left[c_0 - \lambda_2(0)(1 - r_1) - c_s r_1 + c_h(1 - r_1)\left(\frac{P_C - 1}{\theta} - \tau_{F3}\right)\right]\right\} \exp\{\frac{\theta\, \tau_{F3}}{P_C - 1}\}$.(C8)

τ_{F3} is the entry point of Scenario C(vi), that is, the entry point of the time interval that uses strategy S_L (see Table C2).

Based on the necessary conditions from Table C3 and Table A4, together with Table C1, the conditions for the optimal controls under $d^* = 1 - r_2 = r_1 s$ that need to be met to make the optimal decision in Inventory Stage 2 are provided in Table C4.

Table C4. The conditions for optimal decisions in Inventory Stage 2 under $d^* = 1 - r_2 = r_1 s$.

Scenario	The co-state variable λ_1		The conditions		
	$-\dot{\lambda}_1$	λ_1	λ_2	$\lambda_1 > M_1$	
C(i)	0	constant	$> c_s$	$\lambda_1 > c_p + M_1$	$\lambda_1 < c_p + M_2$
C(ii)	0	constant		$\lambda_1 > c_p + M_1$	$\lambda_1 > c_p + M_2$
C(iii)	M_0	F_4	$> c_s$	$\lambda_1 > c_p + M_1$	$\lambda_1 < c_p + M_2$
C(iv)	M_0	F_4		$\lambda_1 > c_p + M_1$	$\lambda_1 > c_p + M_2$
C(v)	0	constant	$> c_s$		$\lambda_1 < M_2$
C(vi)	$< M_0$	F_3	$> c_s$	$\lambda_1 < c_p + M_2$	$\lambda_1 > M_2$

Where $M_0 = \frac{\theta}{P_C - 1} c_p$, $M_1 = c_0 + c_1 - c_l$, $M_2 = c_0 - c_s$, and F_3 is given in (C8).

$$F_4 = \lambda_1(\tau_{F4}) - \frac{\theta\,(t - \tau_{F4})}{P_C - 1} c_p. \tag{C9}$$

τ_{F4} is the entry point of Scenario C(iii) or C(iv), that is, the entry point of the time interval that uses strategy SP_L or P_L (see Table C2).

Synthesizing the above results derived from $d^* = r_1 s < 1 - r_2$ (Table B2, B4) and $d^* = r_1 s = 1 - r_2$ (Table C2, C4), we get Table C5. It presents the optimal decisions in Inventory Stage 2.

Table C5. The optimal strategy under different r_0 in Inventory Stage 2

r_0	Scenario	The optimal strategy	The conditions for each strategy			
			$-\dot{\lambda}_1$	λ_1	λ_1	λ_2
0	B(iv) B(v)	S_H^0	0	constant	$\lambda_1 < M_1 + c_p$	$> c_l - c_1$
	C(i)	SP_H	0	constant	$c_p + M_1 < \lambda_1 < c_p + M_2$	$> c_s$
	C(ii)	P_H	0	constant	$\lambda_1 > c_p + M_2$	
$0 < r_0 < 1 - r_1$	B(iv)	S_H^0	0	constant	$\lambda_1 < M_1$	$> c_l - c_1$
	B(v)	S_H^r	$< M_0$	F_2	$M_1 < \lambda_1 < c_p + M_1$	$> c_l - c_1$
	C(iii)	SP_L	M_0	F_4	$c_p + M_1 < \lambda_1 < c_p + M_2$	$> c_s$
	C(iv)	P_L	M_0	F_4	$\lambda_1 > c_p + M_2$	
$r_0 > 1 - r_1$	B(iv)	S_H^0	0	constant	$\lambda_1 < M_1$	$> c_l - c_1$
	C(v)	S_L^0	0	constant	$M_1 < \lambda_1 < M_2$	$> c_s$
	C(vi)	S_L	$< M_0$	F_3	$M_2 < \lambda_1 < c_p + M_2$	$> c_s$
	C(iv)	P_L	M_0	F_4	$\lambda_1 > c_p + M_2$	

As indicated in Table C5, the co-state variable λ_1 is a constant or a function that drops over time, with a terminal value $\lambda_1(T - t_0) = 0$ (see (23)). Thus, $\lambda_1 \geq 0$ before the time T. As a result, $\lambda_1 < M_2 < 0$ or $\lambda_1 < M_1$ never appear. Accordingly, the strategies S_H^0 and S_L^0 can be excluded from the case of $r_0 > 1 - r_1$, and S_H^0 can be excluded from $0 < r_0 < 1 - r_1$. We get Table 5. □

Part D:

Derivation of Figures 2-3

(i) For Figure 2

When the inventory remains positive in Inventory Stage 1, transitions between strategies can only occur under two situations: if the conditions defined by co-state variables λ_1 and λ_2 are not met or if the demand state defined by customers' behavior r_0 changes.

(a) The dynamics of λ_1 and λ_2 in Inventory Stage 1

As stated in Table 4, $\lambda_1 = F_1$ under strategies I_L and SI_L and $\lambda_1 = C$ under I_H and SI_H. According to the expression of F_1 (See (A11-A13)), we have (D1-D2) for I_L and SI_L.

$$\lambda_1 + \lambda_2 = C_1 e^{-\frac{\theta \tau_{F1}}{(P_C - 1)}} + c_0 + c_h \frac{P_C - 1}{\theta}. \tag{D1}$$

$$\dot{\lambda_1} + \dot{\lambda_2} = c_h - \frac{\theta}{P_C - 1}(\lambda_1 + \lambda_2 - c_0). \tag{D2}$$

That is, $\lambda_1 + \lambda_2$ increases over time when $\lambda_1 + \lambda_2 < c_0 + c_h(P_C - 1)/\theta$. Therefore, together with $\lambda_2 = \lambda_2(0) + c_h t$, the property (D3) is determined.

λ_2 increases over time;

under strategies I_H and SI_H: $\lambda_1 + \lambda_2 = C + \lambda_2$ increases over time;

under strategies I_L and SI_L: $\lambda_1 + \lambda_2$ increases over time at the first phase where $\lambda_1 + \lambda_2 \in$

$$[c_0, c_0 + c_h(P_C - 1)/\theta]. \tag{D3}$$

Thus, we have the following findings: First, the transition from $\lambda_1 + \lambda_2 > c_0$ to $\lambda_1 + \lambda_2 < c_0$ never happens, and the transition from $\lambda_1 + \lambda_2 < c_0$ to $\lambda_1 + \lambda_2 > c_0$ can be activated (transition ③ in Figure 2). Second, the transition from $\lambda_2 < c_s$ to $\lambda_2 > c_s$ can be activated (transition ④ in Figure 2).

(b) The dynamics of r_0 in Inventory Stage 1

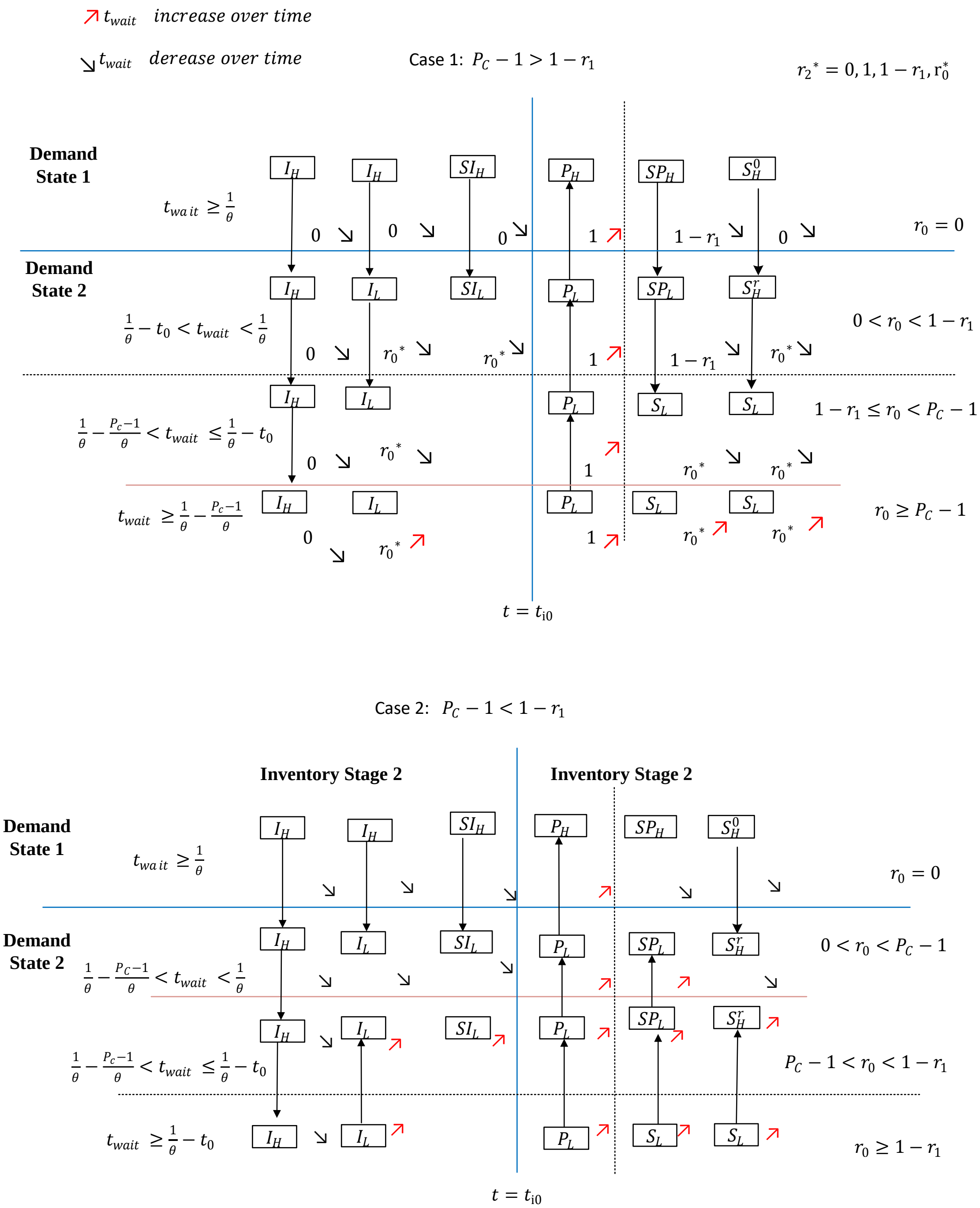


Figure D1. The dynamics of t_{wait} under different strategies.

According to Lemma 1, the required waiting time of customers who arrive at time t increases over time if the backorder rate exceeds $P_C - 1$. As stated in Table 3, r_2^* could be achieved as r_0^*, 1, 0, and $1 - r_1$ under the implementations of our proposed elven types of strategies. Therefore, in order to present the dynamics of customers' behavior, we consider two cases: Case 1 where $P_C - 1 > 1 - r_1$ and Case 2 where $P_C - 1 < 1 - r_1$, and accordingly divide Demand State 2 where $r_0 > 0$ into three sub-states, as shown in Figure D1.

In Inventory Stage 1, $r_2^* = r_0^*$ under strategies I_L and SI_L and $r_2^* = 0$ under I_H and SI_H. The dynamics of customers' waiting time are depicted in Figure D1, which defines when and how the demand state changes over time. The transition from Demand State 1 to Demand State 2 can be activated in Inventory Stage 1 (transition ① in Figure 2).

(ii) For Figure 3

(a) The dynamics of λ_1 in Inventory Stage 2

As stated in Table 5, after the inventory is entirely depleted, the conditions of optimal strategies are defined by λ_1 in Inventory Stage 2, where and $\lambda_1 = C, F_2, F_3, F_4$. According to the expressions of F_2, F_3, F_4 (see Tables B4 and C4), we see that λ_1 decreases over time or remains constant in the time intervals of Inventory Stage 2. Thus, the transitions activated by λ_1 can only occur at two types of time points: from $\lambda_1 < c_p + M_2$ to $\lambda_1 > c_p + M_2$ (transition ⑤ in Figure 3) or from $c_p + M_1 < \lambda_1 < c_p + M_2$ to $\lambda_1 < c_p + M_1$ (transition ⑥ in Figure 3);

(b) The dynamics of r_0 in Inventory Stage 2

Similarly, $r_2^* = r_0^*$, 1, 0, and $1 - r_1$ in Inventory Stage 2, and the dynamics of customers' waiting time are also presented in Figure D1 to define the transition trend of the backlogged demand state (transitions ① and ② in Figure 3).

Summing up, we get Figures 2-3. □

Part E:

Calculation of Table 6

Before examining possible transitions at time t_{i0}, we present the values of the Hamilton function for the proposed eleven types of strategies. Here, for the sake of convenience, we denoted the value of the Hamilton function $H({r_2}^*, s^*, p^*, d^*, b^*, I^*, \lambda_1, \lambda_2, t_{i0})$ for the strategy I_H as $H_1(I_H)$, and similarly for the other strategies.

Recall from the calculation of Tables 4-5 that the strategies I_H, I_L, SI_H and SI_L are derived from the Hamilton function (33), S_H^0 and S_H^r are from (35), and SP_H, SP_L, P_H, and P_L are from (36). Therefore, by substituting $\{s^*, {r_2}^*, p^*, d^*\}$ of each strategy (see Table 3) into the above three Hamilton functions, we get (E1)-(E11).

$$H_1(I_H) = -\lambda_2 + c_0 + c_h I_0. \tag{E1}$$

$$H_1(I_L) = (\lambda_1 + \lambda_2 - c_0) - \lambda_2 + c_0 + c_h I_0. \tag{E2}$$

$$H_2(S_H^0) = (c_l - c_1 - c_s) r_1 - c_l + c_0 + c_1 + c_h I_0. \tag{E3}$$

$$H_2(S_H^r) = (c_l + \lambda_1 - c_0 - c_1) r_0 + (c_l - c_1 - c_s) r_1 - c_l + c_0 + c_1 + c_h I_0. \tag{E4}$$

$$H_3(P_H) = \lambda_1 - c_p + c_h I_0. \tag{E5}$$

$$H_3(P_L) = \lambda_1 - c_p(1 - r_0) + c_h I_0. \tag{E6}$$

$$H_3(SP_H) = \left(\lambda_1 + c_s - c_0 - c_p\right)(1 - r_1) - c_s + c_0 + c_h I_0. \tag{E7}$$

$$H_3(SP_L) = \left(\lambda_1 + c_s - c_0 - c_p\right)(1 - r_1) + c_p\, r_0 - c_s + c_0 + c_h I_0. \tag{E8}$$

$$H_3(S_L) = (\lambda_1 + c_s - c_0)\, r_0 - c_s + c_0 + c_h I_0\ . \tag{E9}$$

$$H_1(SI_H) = (\lambda_2 - c_s)\, r_1 - \lambda_2 + c_0 + c_h I_0. \tag{E10}$$

$$H_1(SI_L) = (\lambda_1 + \lambda_2 - c_0)\, r_0 + (\lambda_2 - c_s)\, r_1 - \lambda_2 + c_0 + c_h I_0. \tag{E11}$$

By examining the jump condition (37) for the possible transitions, we exclude the cases that cannot be met, and present the conditions that cannot be met directly. Note that, due to the continuity of the backlogged demand b, we have $r_0({t_{i0}}^-) = r_0({t_{i0}}^+) = r_0(t_{i0})$ and $\lambda_1({t_{i0}}^+) = \lambda_2({t_{i0}}^-)$. The results are summarized in Table E1.

(i) $\boldsymbol{I_H - S_H^0}$: ${\lambda_2}^- = c_s r_1 + (c_l - c_1)(1 - r_1)$. Thus, the transition is excluded based on ${\lambda_2}^- \leq c_s$.

(ii) $\boldsymbol{I_H - S_H^r}$: ${\lambda_2}^- = -({\lambda_1}^+ - c_0) r_0({t_{i0}}^-) + c_s r_1 + (c_l - c_1)\left(1 - r_1 - r_0({t_{i0}}^-)\right)$. The transition happens if and only if ${\lambda_2}^- = c_s$, ${\lambda_1}^+ = c_0 - c_s$, and $r_0({t_{i0}}^+) = 1 - r_1$. Thus, this transition is excluded.

(iii) $\boldsymbol{I_H - SP_H}$**:** $\lambda_2{}^- - c_s = [c_p - (\lambda_1{}^+ + c_s - c_0)](1 - r_1)$ is required. However, due to $\lambda_1{}^+ \le c_0 - c_s + c_p$ under SP_H and $\lambda_2{}^- \le c_s$, the equation happens if and only if $\lambda_2{}^- = c_s$ and $\lambda_1{}^+ = -c_s + c_0 + c_p$.

(iv) $\boldsymbol{I_H - SP_L}$ **:** $\lambda_2{}^- - c_s + c_p r_0(t_{i0}{}^+) - (c_p + c_0 - c_s - \lambda_1{}^+)(1 - r_1) = 0$. Due to $r_0(t_{i0}{}^+) \le 1 - r_1$ under SP_H and $\lambda_2{}^- \le c_0 - \lambda_1{}^-$under I_H, we have $0 = \lambda_2{}^- - c_s + c_p r_0(t_{i0}{}^+) - (c_p + c_0 - c_s - \lambda_1{}^+)(1 - r_1) \le \lambda_2{}^- - c_s - (c_0 - c_s - \lambda_1{}^+)(1 - r_1)$, that is $\lambda_2{}^- - c_s \ge (c_0 - c_s - \lambda_1{}^+)(1 - r_1)$. On the other hand, $\lambda_2{}^- \le c_0 - \lambda_1{}^- = c_0 - \lambda_1{}^+$ under I_H. Thus, we have $\lambda_2{}^- - c_s \ge (\lambda_2{}^- - c_s)(1 - r_1)$. The inequality happens if and only if $\lambda_2{}^- = c_s$. Accordingly, $\lambda_1{}^+ = c_0 - c_s$ and $r_0(t_{i0}{}^+) = 1 - r_1$ are determined based on the jump condition. The transition is excluded based on $\lambda_1{}^+ \ge 0$.

(v) $\boldsymbol{I_H - P_L}$**:** $\lambda_2{}^- = c_p\,[1 - r_0(t_{i0}{}^+)] + c_0 - \lambda_1{}^+$. Thus, $\lambda_1{}^- + \lambda_2{}^- = \lambda_1{}^+ + \lambda_2{}^- > c_0$. The transition is excluded.

(vi) $\boldsymbol{I_H - S_L}$**:** $\lambda_2{}^- + (\lambda_1{}^+ + c_s - c_0)r_0(t_{i0}{}^+) - c_s = 0$. Based on $\lambda_1{}^- - c_0 \le -\lambda_2{}^-$, we have $0 \le (\lambda_2{}^- - c_s)(1 - r_0(t_{i0}{}^+))$. Similarly, $\lambda_1{}^+ = c_0 - c_s$ and $\lambda_2{}^- = c_s$ is achieved. The transition is excluded based on $\lambda_1{}^+ \ge 0$.

(vii) $\boldsymbol{I_L - SP_L}$: $\lambda_2{}^-(1 - r_0(t_{i0}{}^+)) + (\lambda_1{}^+ - c_0 - c_p)(1 - r_1 - r_0(t_{i0}{}^+)) - c_s r_1 = 0$. The transition happens if and only if $\lambda_2{}^- = c_s$ and $r_0(t_{i0}{}^+) = 1 - r_1$.

(viii) $\boldsymbol{I_L - S_L}$**:** $\lambda_2{}^- = (c_s - \lambda_2{}^-)\,r_0(t_{i0}{}^+) + c_s$, that is, $\lambda_2(t_{i0}{}^-) = c_s$.

(ix) $\boldsymbol{I_L - S_H^r}$**:** $\lambda_2{}^-[1 - r_0(t_{i0}{}^-)] = c_s r_1 + (c_l - c_1)(1 - r_1 - r_0(t_{i0}{}^-))$. The inequality happens if and only if $\lambda_2{}^- = c_s$ and $r_0(t_{i0}{}^-) = 1 - r_1$.

(x) $\boldsymbol{SI_H - P_H}$**:** $\lambda_2{}^-(1 - r_1) - (c_p + c_0 - c_s r_1) + \lambda_1{}^+ = 0$. Due to $\lambda_2{}^- > c_s$ and $\lambda_1{}^+ \ge c_p + c_0 - c_s$, we see that $0 \ge c_s(1 - r_1) - (c_p + c_0 - c_s r_1) + c_p + c_0 - c_s$. The transition is excluded.

(xi) $\boldsymbol{SI_H - P_L}$: The transition requires $c_0 - r_1 c_s + c_p(1 - r_0) = \lambda_1{}^+ + \lambda_2{}^-(1 - r_1) \ge c_p + c_0 - c_s + (1 - r_1)c_s$. The inequality cannot happen.

(xii) $\boldsymbol{SI_H - SP_L}$: $(\lambda_1{}^- + \lambda_2{}^- - c_0 - c_p)(1 - r_1) + c_p\,r_0(t_{i0}{}^+) = 0$. The transition is excluded because of $\lambda_1{}^- + \lambda_2{}^- - c_0 - c_p < 0$.

(xiii) $\boldsymbol{SI_H - S_L}$: $(\lambda_1{}^{-} + c_s - c_0)\, r_0(\,t_{i0}{}^{+}) + (\lambda_2{}^{-} - c_s)(1 - \,r_1) = 0$. Due to $\lambda_1{}^{-} - c_0 \le -\lambda_2{}^{-}$, the following inequality is deduced: $(\lambda_2{}^{-} - c_s)\big(1 - \,r_1 - r_0(\,t_{i0}{}^{+})\big) \ge 0$. Note that $\big(1 - \,r_1 - r_0(\,t_{i0}{}^{+})\big) \le 0$ under S_L and $\lambda_2{}^{-} - c_s \ge 0$ under SI_H. The transition is excluded.

(xiv) $\boldsymbol{SI_H - S_H^r}$: $\big(c_l + \lambda_1{}^{+} - c_0 - c_1\big) r_0(\,t_{i0}{}^{-}) + (\lambda_2{}^{-} - c_l + c_1)(1 - \,r_1) = 0$. The transition happens if and only if $\lambda_1{}^{+} + \lambda_2{}^{-} = c_0$. Accordingly, $(\lambda_2{}^{-} - c_l + c_1)\big(1 - r_1 - r_0(\,t_{i0}{}^{-})\big) = 0$ is determined. On the other hand, $\lambda_1{}^{+} = c_0 - c_l + c_1 < 0$ is achieved if $\lambda_2{}^{-} = c_l - c_1$. Therefore, $1 - \,r_1 - r_0(\,t_{i0}{}^{-}) = 0$ has to be met.

(xv) $\boldsymbol{SI_L - P_L}$: $\lambda_2{}^{-}[\,1 - r_0(\,t_{i0}{}^{-}) - r_1] = -r_1 c_s + (c_p + c_0 - \lambda_1{}^{+})[\,1 - r_0(\,t_{i0}{}^{-})]$. The transition is excluded.

Table E1. The jump conditions for possible transitions at time t_{i0}.

Transitions	The jump condition for each transition	
$I_H - S_H^0$		Excluded
$I_H - SP_H$	$\lambda_2{}^{-} = c_s$ and $\lambda_1{}^{+} = -c_s + c_0 + c_p$	
$I_H - P_H$	$\lambda_2{}^{-} = c_p + c_0 - \lambda_1{}^{+}$	
$I_H - S_H^r$		Excluded
$I_H - P_L$		Excluded
$I_H - SP_L$		Excluded
$I_H - S_L$		Excluded
$I_L - P_L$	$\lambda_2{}^{-} = c_p + c_0 - \lambda_1{}^{+}$	
$I_L - SP_L$	$\lambda_2{}^{-} = c_s$ and $r_0(\,t_{i0}{}^{+}) = 1 - r_1$	
$I_L - S_L$	$\lambda_2{}^{-} = c_s$	
$I_L - S_H^r$	$\lambda_2{}^{-} = c_s$ and $r_0(\,t_{i0}{}^{+}) = 1 - r_1$	
$SI_H - S_H^0$	$\lambda_2{}^{-} = c_l - c_1$	
$SI_H - P_H$		Excluded
$SI_H - P_L$		Excluded
$SI_H - SP_H$	$\lambda_2{}^{-} = -\lambda_1{}^{+} + c_0 + c_p$	
$SI_H - SP_L$		Excluded
$SI_H - S_L$		Excluded
$SI_H - S_H^r$	$\lambda_1{}^{+} + \lambda_2{}^{-} = c_0$ and $r_0(\,t_{i0}{}^{+}) = 1 - r_1$	
$SI_L - S_H^r$	$\lambda_2{}^{-} = c_l - c_1$	
$SI_L - SP_L$	$\lambda_2{}^{-} = c_p + c_0 - \lambda_1{}^{+}$	
$SI_L - P_L$		Excluded

where, $\lambda_2(\,t_{i0}{}^{-})$, $\lambda_1(\,t_{i0}{}^{+})$ and $\lambda_1(\,t_{i0}{}^{-})$ are denoted as $\lambda_2{}^{-}$, $\lambda_1{}^{+}$ and $\lambda_1{}^{-}$.

Table 6 is achieved. □

Proof of Corollary 1

If the customers' reaction always remains at the state of $r_0 = 0$, we have

$$r_0(T - t_0) = [1 - \theta \frac{b(T-t_0)}{P_C - 1}]^+ = 0, r_0(t_{i0}) = [1 - \theta \frac{b(t_{i0})}{P_C - 1} - \theta T + \theta t_{i0}]^+ = 0,$$

$$\text{and } r_0(0) = [1 - \theta T]^+ = 0. \quad \text{(E12)}$$

According to Table 4, no backorder occurs at Inventory Stage 1 if $r_0 = 0$. Therefore, $b(t_{i0}) = 0$, and (E12) equals to

$$1 - \theta \frac{b(T-t_0)}{P_C - 1} \leq 0 \text{ and } 1 - \theta T + \theta t_{i0} \leq 0. \quad \text{(E13)}$$

(i) About $I_H - SI_H - SP_H$:

As indicated in Figures 3-5, the optimal strategy is determined as $I_H - SI_H - SP_H$ under $r_0 = 0$, if it is optimal to start with I_H and terminates with SP_H. That is,

$$\lambda_2(0) < c_s, \text{ and } c_s < c_p + c_0 < c_l - c_1. \quad \text{(E14)}$$

Next, we identify the terminal quantity of the backlogged demand and the time point t_{i0}, along with the initial value $\lambda_2(0)$ of the co-state variable λ_2.

First, as indicated in f_3, the backlogged demand grows at the rate of $1 - r_1$ in Inventory Stage 2. Hence, the terminal quantity of the backlogged demand is determined as

$$b(T - t_0) = (1 - r_1)(T - t_0 - t_{i0}). \quad \text{(E15)}$$

Substituting (E15) into (E13), we have

$$T \geq \frac{P_C - 1}{\theta^2 t_0} + t_{i0} + t_0, \text{ and } T \geq \frac{1}{\theta} + t_0 + t_{i0}. \quad \text{(E16)}$$

Second, observing the inventory consumption speed in the process of using I_H and SI_H, the time point t_{i0} is identified by the following equation:

$$I_0 = t_s + (1 - r_1)(t_{i0} - t_s). \quad \text{(E17)}$$

Where t_s is given by

$$\lambda_2(t_s) = \lambda_2(0) + c_h t_s = c_s. \quad \text{(E18)}$$

Last, according to the transition $SI_H - SP_H$ at time t_{i0} (Table 6), we have $\lambda_2(t_{i0}^-) = -\lambda_1(t_{i0}) + c_0 + c_p$, where, the co-state variable λ_2 is defined as $\lambda_2 = \lambda_2(0) + c_h t$, and the co-state variable λ_1 is a constant if $r_0 = 0$. Together with $\lambda_1(T - t_0) = 0$, we have $\lambda_2(0) + c_h t_{i0} = c_0 + c_p$, that is,

$$\lambda_2(0) = c_0 + c_p - c_h t_{i0}. \quad \text{(E19)}$$

Based on (E17)-(E19), we have

$$t_s = I_0 - \theta t_0 \frac{c_0+c_p-c_s}{c_h},\ t_{i0} = I_0 + (1-\theta t_0)\frac{c_0+c_p-c_s}{c_h}, \text{ and}$$

$$\lambda_2(0) = \theta t_0\left(c_0 + c_p\right) + (1-\theta t_0)c_s - c_h I_0. \tag{E20}$$

Substituting (E20) into (E14) and (E16), Corollary 1(i) is proved.

(ii) About $\boldsymbol{SI_H - SP_H}$:

Similarly, the optimal strategy is determined as $SI_H - SP_H$ under (E16), if

$$\lambda_2(0) > c_s, \text{ and } c_s < c_p + c_0 < c_l - c_1. \tag{E21}$$

According to the inventory consumption speed when using SI_H, the time point t_{i0} is identified as

$$I_0 = (1-r_1)t_{i0}. \leftrightarrow\ t_{i0} = \frac{I_0}{\theta t_0}. \tag{E22}$$

Substituting (E19) and t_{i0} into (E16) and (E21), Corollary 1(ii) is proved.

(iii) About $\boldsymbol{I_H - P_H}$:

The optimal strategy is determined as $I_H - P_H$ under (E13) if it is optimal to start with I_H and terminates with P_H. That is,

$$\lambda_2(0) < c_s, \text{ and } c_s > c_p + c_0. \tag{E23}$$

According to the transition $I_H - P_H$ at time t_{i0} (see Table 6), $\lambda_2(0)$ is achieved as (E19), where t_{i0} is given as

$$t_{i0} = I_0. \tag{E24}$$

On the other hand, the backlogged demand $b(T-t_0)$ derived from pure compensation P_H is given as

$$b(T) = T - t_0 - t_{i0}. \tag{E25}$$

By substituting (E24) - (E25) into (E13) and (E23), Corollary 1(iii) is achieved.

(iv) About $\boldsymbol{I_H - SI_H - S_H^0}$ or $\boldsymbol{SI_H - S_H^0}$:

No backlogged demand appears in Inventory Stages 1-2. Hence, the terminal quantity of the backlogged demand is determined as $b(T-t_0) = 0$, that is, $t_{wait}\,(T-t_0) = 0$. The optimal strategy $I_H - SI_H - S_H^0$ can be excluded from $r_0 = 0$. □

报告编号：HDJS1420210879

文献检索报告

委托人： 李姗姗

检索数据库：

1. SCI-E 美国《科学引文索引》
2. SSCI 美国《社会科学引文索引》
3. JCR 期刊引证数据库
4. 中科院 JCR 期刊分区数据平台

检索结果：

1. SCI-E 美国《科学引文索引》收录论文 6 篇；
2. SSCI 美国《社会科学引文索引》收录论文 3 篇；
3. 其他详细信息请见附件。

检索日期： 2021 年 11 月 26 日

声明： 本报告检索的文献信息均由委托人提供并确认，如果由于委托人提供信息不实而造成任何后果，本中心概不负责。

检索人（签名）：周建

湖南大学图书馆科技查新与文献检索中心

作者: Li, SS (Li, Shanshan);He, Y (He, Yong);Minner, S (Minner, Stefan)
来源出版物: INTERNATIONAL JOURNAL OF PRODUCTION RESEARCH **卷:** 59 **期:** 5 **页:** 1511-1533 **出版年:** MAR 4 2021 **DOI:** 10.1080/00207543.2020.1840643
Web of Science 核心合集中的 "被引频次": 4
入藏号: WOS:000590101400001
文献类型: Article
地址:[Li, Shanshan] Nanjing Audit Univ, Sch Finance, Nanjing, Peoples R China.
[Li, Shanshan; He, Yong] Southeast Univ, Sch Econ & Management, Nanjing 210096, Peoples R China.
[Minner, Stefan] Tech Univ Munich, TUM Sch Management, Munich, Germany.
通讯作者地址: He, Y (通讯作者)， Southeast Univ, Sch Econ & Management, Nanjing 210096, Peoples R China.
电子邮件地址: hy@seu.edu.cn
Web of Science 类别: Engineering, Industrial; Engineering, Manufacturing; Operations Research & Management Science
ISSN:0020-7543
eISSN:1366-588X

2020 年公布的影响因子：8.568, JCR 期刊引证数据库分区（2020）

JCR * 类别	类别中的排序	JCR 分区
ENGINEERING, INDUSTRIAL	4/49	Q1
ENGINEERING, MANUFACTURING	3/50	Q1
OPERATIONS RESEARCH & MANAGEMENT SCIENCE	2/84	Q1

中科院 JCR 期刊分区及影响因子（2020）：

期刊全称	INTERNATIONAL JOURNAL OF PRODUCTION RESEARCH					
期刊简称	INT J PROD RES			ISSN	0020-7543	
年份	2020			综述	否	
	学科名称				分区	Top 期刊
小类	ENGINEERING, INDUSTRIAL 工程：工业				3	-
小类	ENGINEERING, MANUFACTURING 工程：制造				3	-
小类	OPERATIONS RESEARCH & MANAGEMENT SCIENCE 运筹学与管理科学				2	-
大类	管理科学				2	-
期刊影响因子				总被引频次		
2017 年	2018 年	2019 年	2017-2019 年平均	2018 年	2019 年	2018-2019 年平均
2.623	3.199	4.577	3.466	17976	21149	39125

输出日期: 2021-11-26